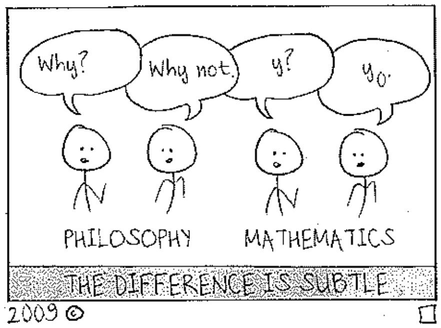
KEY

Math 225: Quiz the Sixth

October 28, 2011

This quiz is closed book and closed notes. Please justify all of your answers. You have 40 minutes.



1. Find a function $f(x, y)$ which satisfies the following equations, or explain why no such function exists.

   (a) $f_x = x^2 + 2y$, $f_y = -2x + 3y^2$

   $$\left.\begin{array}{l} f_{xy} = 2 \\ f_{yx} = -2 \end{array}\right\} \text{no such fcn exists.}$$

   (b) $f_x = e^x + 3y^2$, $f_y = 6xy + \cos(y)$

   $$\left.\begin{array}{l} f_{xy} = 6y \\ f_{yx} = 6y \end{array}\right\} \text{C's theorem holds!}$$

   $f_x = e^x + 3y^2 \qquad f = e^x + 3xy^2 + g(y)$

   $f_y = 6xy + \cos y \qquad f = 3xy^2 + \sin y + g(x)$

   $f(x,y) = e^x + 3xy^2 + \sin y$

2. Find the tangent plane to $f(x,y) = \sqrt{x^2+y^2}$ at the point $(3,-4)$ and use it to approximate $f(3.02, -3.99)$. (No calculators, please!)

Tplane: $f(3,-4) = 25$

$f_x = \frac{x}{\sqrt{x^2+y^2}}$ $\quad f_y = \frac{y}{\sqrt{x^2+y^2}}$

$f_x(3,-4) = \frac{3}{5}$ $\quad f_y(3,-4) = \frac{-4}{5}$

$z = 25 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4)$

Approx

$z \approx 25 + \frac{3}{5}(.02) - \frac{4}{5}(.01)$

$= 25 + \frac{.06}{5} - \frac{.04}{5}$

$= 25 + \frac{.02}{5}$

$25 + \frac{2}{500} = 25.004$

3. Wheat production (W) is dependent on temperature (T) and rainfall (R). It is estimated that temperature is increasing at a rate of 0.05° C/year and rainfall is decreasing at a rate of 0.1 cm/year. It is also estimated that, at current levels of production, $\frac{\partial W}{\partial T} = -1.5$ and $\frac{\partial W}{\partial R} = 4$.

(a) Give a prose description of the significance of the signs of the partial derivatives $\frac{\partial W}{\partial T}$ and $\frac{\partial W}{\partial R}$.

$\frac{\partial w}{\partial T} = -1.5 < 0$ as temperature increases, Wheat production decreases.

$\frac{\partial W}{\partial R} = 4 > 0$ as rainfall increases, wheat production increases.

(b) Using these estimates, find $\frac{dW}{dt}$.

$$\frac{dW}{dt} = \frac{\partial w}{\partial T}\frac{dT}{dt} + \frac{\partial W}{\partial R}\frac{dR}{dt}$$

$= (-1.5)(.05) + (4)(-.1) = -.4 - .075 =$ ~~-4.075~~ bushels/year

$-.475$ bushels/year

$f(x,y)$

4. Let $\cancel{f(x)}$ $= 3x^2 + 2y - 4xy$. Find $D_{\mathbf{u}}f$ at the point (1,1) as we move towards $(6, -11)$.

$$\vec{\nabla} f = \langle 6x - 4y, 2 - 4x \rangle \Big|_{(1,1)} = \langle 6, -2 \rangle$$

$$\vec{u} \text{ from } (1,1) \text{ to } (6,-11) = \frac{\langle 5, -12 \rangle}{|\langle 5, -12 \rangle|} = \left\langle \frac{5}{\sqrt{3}}, \frac{-12}{13} \right\rangle$$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \left\langle \frac{5}{13}, \frac{-12}{13} \right\rangle \cdot \langle 6, -2 \rangle$$

$$= \frac{30}{13} + \frac{24}{13} = \frac{54}{13}.$$

5. Consider the surface $xy^3 + xyz + yz^2 = 16$.

(a) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and give conditions for which ~~theses~~ these partials exist.

$$F = xy^3 + xyz + yz^2 - 16 = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \cancel{\frac{-(3xy^2 + yz)}{xy + 2yz}} \quad \frac{-(y^3 + yz)}{xy + 2yz}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(3xy^2 + xz + z^2)}{xy + 2yz}$$

These partials exist provided

$xy + 2yz \neq 0$

(b) Find the tangent plane to this surface at the point $(-1, 2, 4)$

$$\vec{n} = \vec{\nabla} F = \langle y^3 + yz, \; 3xy^2 + xz + z^2, \; xy + 2yz \rangle \Big|_{(-1,2,4)}$$

$$= \langle 16, 0, 14 \rangle$$

Tplane: $16(x+1) + 0(y-2) + 14(z-4) = 0$

6. (Bonus) A well-behaved function $f(x, y)$ has four second order partial derivatives. Clairaut's theorem guarantees that two of these are the same, so we effectively have three distinct derivatives.

   (a) How many third order derivatives does $f(x, y)$ have? How many are distinct?

   8

   $f_{\_\,\_\,\_}$

   two choices for each

   4

   f_{xxx}

   f_{xxy}

   f_{xyy}

   f_{yyy}

   (b) How many fourth order derivatives does $f(x, y)$ have? How many are distinct?

   16

   5

   f_{xxxx}

   f_{xxxy}

   f_{xxyy}

   f_{xyyy}

   f_{yyyy}

   (c) How many nth order derivatives does $f(x, y)$ have? How many are distinct?

   2^n

   $n+1$