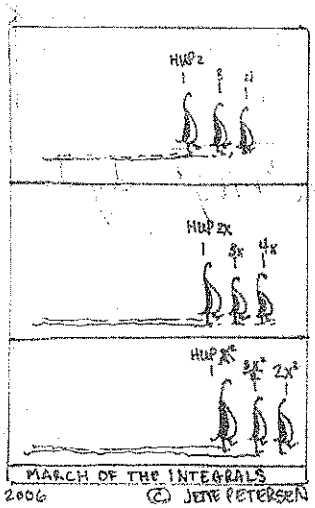
KEY

Math 225: Quiz the ~~Seventh~~ Eighth

December 2, 2011

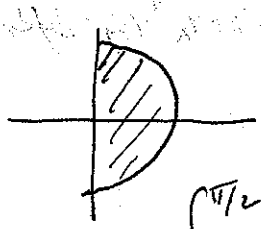
This quiz is closed book and closed notes. Please justify all of your answers. You have 40 minutes.



1. Convert

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x + y \, dy \, dx$$

to polar coordinates, and solve the integral.



$$\int_{-\pi/2}^{\pi/2} \int_0^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^3}{3}(\cos\theta + \sin\theta)\Big|_0^2 \, d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} (\cos\theta + \sin\theta)\, d\theta = \frac{8}{3}\left[\sin\theta - \cos\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{8}{3}(1 - 0 - (-1 - 0))$$

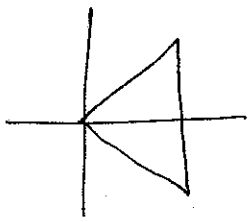
$$= \frac{16}{3}$$

2. Set up the integral to find the area of the surface of $f(x, y) = x^2 + 2y^2$ over the triangle with vertices $(0, 0), (1, 1)$ and $(1, -1)$. You need not compute this integral.

$$S.A. = \iint \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

$f_x = 2x$

$f_y = 4y$



$$\int_0^1 \int_{-x}^{x} \sqrt{1+4x^2+16y^2}\, dy\, dx$$

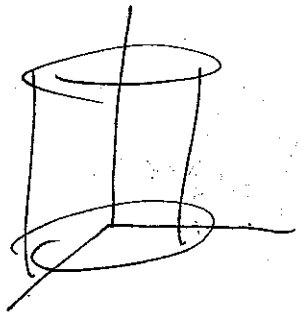
3. Should you get the same integral for the area of the surface of $f(x, y) = x^2 + 2y^2 + 3$? Explain both geometrically and algebraically.

Yes. $f(x,y) = x^2 + 2y^2 + 3$ has the same f_x and f_y.

$$\int_0^1 \int_{-x}^{x} \sqrt{1+4x^2+16y^2}\, dy\, dx$$

Adding a constant only moves the surface up, it does not change the area.

4. Find the volume of a cylinder of radius R and height H using a triple integral in cylindrical coordinates. (Use equations for the standard cylinder and a plane height H above the xy-plane.)

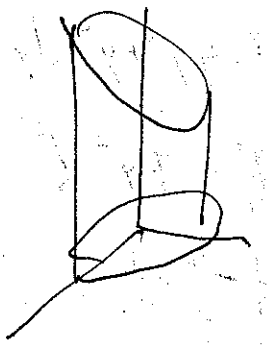


$$\int_0^{2\pi}\int_0^{R}\int_0^{H} 1\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{R} Hr\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{HR^2}{2}\,d\theta = \pi R^2 H.$$

5. Find the volume bound by the cylinder $x^2+y^2=R^2$ and the planes $z=0$ and $z=ax+by+H$ (Here, a and b are chosen so that the plane lies entirely above the $x-y$ plane within the cylinder. This is the standard cylinder, but with a slanted top).



$$\int_0^{2\pi}\int_0^{R}\int_0^{ar\cos\theta+br\sin\theta+H} 1\cdot r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{R} ar^2\cos\theta + br^2\sin\theta + Hr\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{aR^3}{3}\cos\theta + \frac{bR^3}{3}\sin\theta + \frac{HR^2}{2}\,d\theta$$

$$= \frac{ar^3}{3}\sin\theta - \frac{bR^3}{3}\cos\theta + \frac{HR^2}{2}\theta\Big|_0^{2\pi} = 0-0+\frac{HR^2}{2}2\pi$$

$$= \pi HR^2$$

6. Comment on your answers to these two questions.

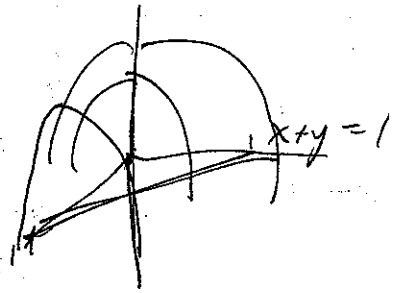
The answers are the same. The slant has no net effect on the volume of the cylinder. (what is subtracted is added and v.v. versa)

7. Evaluate

$$\iiint_E y \, dV$$

where E is bound by $z = 4 - y^2$, $x + y = 1$ and the three coordinate planes.

$$\int_0^1 \int_0^{1-y} \int_0^{4-y^2} y \, dz \, dx \, dy$$



$$= \int_0^1 \int_0^{1-y} \frac{(4-y^2)^2}{2} \, dx \, dy$$

$$= \int_0^1 \int_0^{1-y} \frac{16 - 8y^2 + y^4}{2} \, dx \, dy$$

$$= \frac{1}{2} \int_0^1 (1-y)(16 - 8y^2 + y^4) \, dy$$

$$= \frac{1}{2} \int_0^1 16 - 8y^2 + y^4 - 16y + 8y^3 - y^5 \, dy$$

$$= \frac{1}{2} \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} - 8y^2 + 2y^4 - \frac{y^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[16 - \frac{8}{3} + \frac{1}{5} - 8 + 2 - \frac{1}{6} \right]$$

$$= \frac{1}{2} \left[8 - \frac{2}{3} + \frac{1}{30} \right] = \frac{1}{2} \left[\frac{221}{30} \right] = \frac{221}{60}$$

$$\int_0^1 \int_0^{1-y} y(4-y^2) \, dx \, dy$$

$$= \int_0^1 \int_0^{1-y} 4y - y^3 \, dx \, dy$$

$$= \int_0^1 (4y - y^3)(1-y) \, dy$$

$$= \int_0^1 4y - y^3 - 4y^2 + y^4 \, dy$$

$$= \left[2y^2 - \frac{y^4}{4} - \frac{4}{3}y^3 + \frac{y^5}{5} \right]_0^1$$

$$= 2 - \frac{1}{4} - \frac{4}{3} + \frac{1}{5}$$

$$\frac{2}{3} - \frac{1}{4} + \frac{1}{5}$$

$$\frac{40 - 15 + 12}{60} = \frac{37}{60}$$

8. (Bonus) You may give either $\frac{1}{2}$ extra credit points to yourself, or none to yourself and $\frac{1}{20}$ to each other member of the class. Your total EC will be the sum of what you take for yourself and what others give to you.