KEY

**Math 225: Exam the First** October 13, 2017

You have two hours to take this closed-book, closed-note, and closed-colleague exam. You may use a basic calculator for arithmetic, trig functions, logarithms and exponentials, but no graphing or calculus functions.

1. Consider the parametric curve given by

$$x(t) = t^2 - 2t \quad y(t) = t + 1$$

(a) Where (what points) does the curve cross the $x$- and $y$-axis?

$y$-axis , $x = 0 = t^2 - 2t$ $\quad t = 0, 2 \quad y = 1, 3 \quad$ points $(0,1)$ $(0,3)$

$x$-axis $\quad y = 0 = t + 1 \quad t = -1 \quad x = (-1)^2 - 2(-1) = 3 \quad$ Point $(3,0)$

(b) Where (if anywhere) does the curve have a horizontal or vertical tangent line?

H-Tan $\quad \frac{dy}{dt} = 0 = 1$ , none.

V-Tan $\quad \frac{dx}{dt} = 0 = 2t - 2$ , $t = 1 \quad$ point: $(-1, 2)$

(c) Eliminate the parameter to give $x$ as a function of $y$, and give a general description of the curve.

$x = t^2 - 2t$

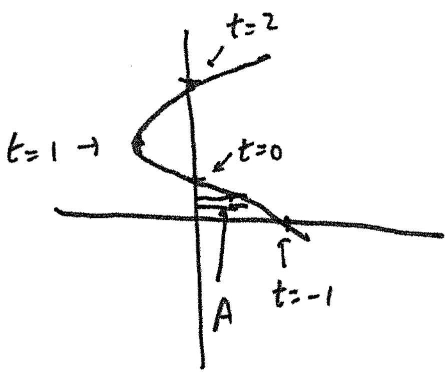
$y = t + 1$

$t = y - 1$

parabola opening right

$x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3$

(d) Find the area in the first quadrant bound by the curve and the coordinate axes.




$$A = \int_{-1}^{0} x \, dy$$

$$= \int_{-1}^{0} (t^2 - 2t)(1) \, dt$$

$$= \frac{t^3}{3} - t^2 \Big]_{-1}^{0} = 0 - \left[\frac{-1}{3} - 1\right]$$

$$= 4/3$$

2. (a) Find the equation of the line perpendicular to the plane $4x + y - 2z = 6$ and through the point $(8, 2, 6)$.

line needs $\vec{v} = \langle 4,1,-2 \rangle$

point: $(8,2,6)$

line: $\langle 8,2,6 \rangle + t\langle 4,1,-2 \rangle = \vec{l}(t)$ .

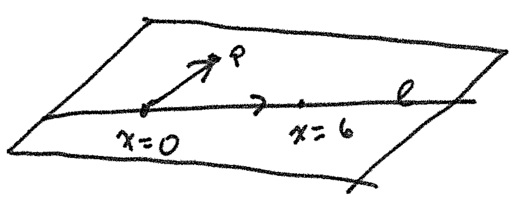
(b) Find the points at which this line intersects each of the coordinate planes.

$(yz)$ $x = 8+4t = 0$ $\quad t=-2 \rightarrow (0,0,10)$

$(xz)$ $y = 2+t = 0$ $\quad t=-2 \rightarrow (0,0,10)$

$(xy)$ $z = 6-2t = 0$ $\quad t=3 \rightarrow (20,5,0)$

3. Find the equation of the plane through the point (2,-2,0) which contains the line $x = 3y = 2z$ (Hint: you can find three points on this plane and go from there).




points $(2,-2,0)$

$(0,0,0)$

$(6,2,3)$

$\vec{v_1} = \langle 2,-2,0 \rangle$

$\vec{v_2} = \langle 6,2,3 \rangle$

$\vec{n} = \langle -6,-6,16 \rangle$

plane: $-6(x-2)-6(y+2)+16z = 0$ $\quad$ or $16z = 6x+6y$ .

4. Suppose that $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 5$. (Give geometric answers for the second part of each of these).

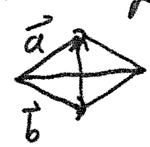
   (a) The maximum value of $\mathbf{a} \cdot \mathbf{b}$ is $\underline{15}$ and occurs when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

   (b) The maximum value of $|\mathbf{a} \times \mathbf{b}|$ is $\underline{15}$ and occurs when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

   (c) The maximum value of $|\mathbf{a} + \mathbf{b}|$ is $\underline{8}$ and occurs when $\mathbf{a}$ and $\mathbf{b}$ are parallel. ~~(or anti parallel)~~

5. Suppose that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal vectors. Prove that $|\mathbf{a}| = |\mathbf{b}|$.

rhombus!



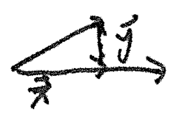

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 0$$

$$|\vec{a}|^2 - |\vec{b}|^2 = 0 \quad \rightarrow \quad |\vec{a}|^2 = |\vec{b}|^2$$

$$\rightarrow |\vec{a}| = |\vec{b}|$$

6. Write the vector $\langle -1, 5, 3 \rangle$ as a sum of a vector *parallel* to $\langle 4, 2, 4 \rangle$ and a vector *perpendicular* to $\langle 4, 2, 4 \rangle$




$$\vec{x} = \mathrm{proj}_{\langle 4,2,4 \rangle} \langle -1, 5, 3 \rangle = \frac{\langle 4,2,4 \rangle \cdot \langle -1,5,3 \rangle}{|\langle 4,2,4 \rangle|^2} \langle 4, 2, 4 \rangle$$

$$= \frac{-4 + 10 + 12}{4^2 + 2^2 + 4^2} \langle 4, 2, 4 \rangle = \langle 2, 1, 2 \rangle$$

$$\vec{y} = \langle -1, 5, 3 \rangle - \vec{x} = \langle -1, 5, 3 \rangle - \langle 2, 1, 2 \rangle = \langle -3, 4, 1 \rangle$$



$$\langle -1, 5, 3 \rangle = \langle 2, 1, 2 \rangle + \langle -3, 4, 1 \rangle$$

7. Consider $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 3 - 4\cos(t)\rangle$

(a) This curve is the intersection of which *cylinder* and which *plane*?

$(2\cos(t))^2 + (2\sin(t))^2 = 4$

$x^2 + y^2 = 4$ ← Cylinder

$z = 3 - 2x$ ← plane

(b) What are maximum and minimum allowable values for each of $x$, $y$, and $z$?

$-2 \leq x \leq 2$

$-2 \leq y \leq 2$

$-1 \leq z \leq 7$

(c) Find $\mathbf{T}(\frac{\pi}{2})$ for this curve.

$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos t, 4\sin(t) \rangle$

$\vec{r}\,'(\pi/2) = \langle -2, 0, 4 \rangle$

$T(\pi/2) = \dfrac{\vec{r}\,'(\pi/2)}{|\vec{r}\,'(\pi/2)|} = \left\langle \dfrac{-2}{\sqrt{20}}, 0, \dfrac{4}{\sqrt{20}} \right\rangle$

$= \left\langle \dfrac{-1}{\sqrt{5}}, 0, \dfrac{2}{\sqrt{5}} \right\rangle$

(d) Find the equation of the tangent line to this curve at $t = \dfrac{\pi}{2}$.

$\vec{\ell}(t) = \vec{r}(\pi/2) + t\,\vec{r}\,'(\pi/2)$

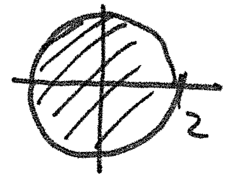
$= \langle 0, 2, 3 \rangle + t \langle -2, 0, 4 \rangle.$

8. Suppose that $f(x, y) = \sqrt{16 - 4x^2 - 4y^2}$.

(a) Find and sketch the domain of $f$.

$16 - 4x^2 - 4y^2 \geq 0$

$\rightarrow 16 \geq 4x^2 + 4y^2 \qquad x^2 + y^2 \leq 4$



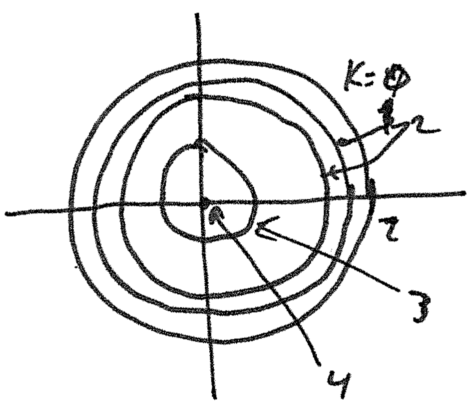

(b) For which values $k$ can we draw level curves for $f$?

$0 \leq \sqrt{16 - 4x^2 - 4y^2} \leq 4$

$\uparrow$

Max when $x = y = 0$

(c) Sketch level curves for $k = 0, 1, 2, 3, 4$, with attention to spacing.




(d) Find $f_x(x, y)$ and $f_y(x, y)$.

$f_x = \frac{1}{2}(16 - 4x^2 - 4y^2)^{-1/2} \cdot (-8x) = \dfrac{-4x}{\sqrt{16 - 4x^2 - 4y^2}}$

$f_y : \frac{1}{2}(16 - 4x^2 - 4y^2)^{-1/2}(-8y) = \dfrac{-4y}{\sqrt{16 - 4x^2 - 4y^2}}$

(e) The graph of $f$ is the top half of which of our quadratic surfaces?

$z = \sqrt{16 - 4x^2 - 4y^2} \Rightarrow z^2 = 16 - 4x^2 - 4y^2$

$4x^2 + 4y^2 + z^2 = 16$

ellipsoid $\rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$

9. Explain why a straight line has zero curvature. (Your argument should give a formula for curvature at some point).

$$K = \frac{|\vec r\,'(t) \times \vec r\,''(t)|}{|\vec r\,'(t)|^3}$$

But for a line $\vec r\,''(t) = 0$

So $K = 0$

10. Describe the motion of a particle that has $a_T$ nonzero, but $a_N$ zero.

$$a_T = \frac{\vec V \cdot \vec a}{|\vec a|} \qquad a_N = \frac{|\vec v \times \vec a|}{|\vec a|}$$

So $\vec v \times \vec a = 0$ when $\vec v \parallel \vec a$

or when our motion is in a straight line

11. Describe the motion of a particle that has $a_T$ zero, but $a_N$ nonzero.

Here, $\vec v \perp \vec a$, so the motion is along a circle of a fixed radius