

KEY

Math 225: Exam the Second November 8, 2017

You have two hours to complete this closed-book, closed-note, closed colleague exam. You may use a calculator for arithmetic only (trig functions and exponentials are okay, but no plotting and no calculus).

1. Let $f(x, y) = x^2 + 2x + \sqrt{y} + ye^x$.

- (a) Find the tangent plane to $f(x, y)$ at the point $(0, 4)$.

$$f(0, 4) = 0^2 + 2(0) + \sqrt{4} + 4e^0 = 6$$

$$f_x(0, 4) = 2x + 2 + 0 + y e^x \Big|_{(0, 4)} = 2(0) + 2 + 0 + 4e^0 = 6$$

$$f_y(0, 4) = 0 + 0 + \frac{1}{2} y^{-\frac{1}{2}} + e^x \Big|_{(0, 4)} = \frac{1}{2} + e^0 = \frac{5}{4}.$$

Plan: $z = 6 + 6(x - 0) + \frac{5}{4}(y - 4)$

- (b) Use the tangent plane to approximate $f(0.04, 3.98)$.

$$\begin{aligned} f(0.04, 3.98) &\approx 6 + 6(0.04 - 0) + \frac{5}{4}(3.98 - 4) \\ &= 6 + .24 + \frac{5}{4}(-.02) \\ &= 6.215 \end{aligned}$$

2. Let $f(x, y) = \sqrt{x^2 + y^2}$.

(a) Find $\nabla(f)$ at the point $(-5, 12)$.

$$f_x = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle \Big|_{(-5, 12)} = \left\langle \frac{-5}{\sqrt{13}}, \frac{12}{\sqrt{13}} \right\rangle$$

(b) Find $D_u f$ as we move from $(-5, 12)$ towards $(-3, 15)$ (Please leave your answer as a radical).

$$(-5, 12) \rightarrow (-3, 15) = \langle 2, 3 \rangle$$

$$\xrightarrow{\vec{u}} \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle$$

$$\left\langle \frac{-5}{\sqrt{13}}, \frac{12}{\sqrt{13}} \right\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \frac{26}{13\sqrt{13}} = \frac{2}{\sqrt{13}}$$

(c) Argue that, at any point except the origin, $\nabla(f)$ is a unit vector. For a bonus, use the graph of f to explain why.

$$\left| \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle \right| \text{ so long as } (x, y) \neq (0, 0)$$

$$= \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} = \sqrt{\frac{x^2 + y^2}{x^2 + y^2}} = 1$$

3. Let $f(x, y) = 6xy - x^2 - y^3$. Find and classify the critical points of f .

$$\begin{aligned} f_x &= 6y - 2x = 0 \quad \rightarrow \quad x = 3y \\ f_y &= 6x - 3y^2 = 0 \quad 6(3y) - 3y^2 = 0 \\ &\quad 18y - 3y^2 = 0 \\ &\quad 3y(6 - y) = 0 \\ y &= 0 \quad y = 6 \quad y = 0, 6 \\ x &= 0 \quad x = 18 \end{aligned}$$

$D = f_{xx}f_{yy} - (f_{xy})^2$

points, $(0, 0)$ $D < 0$	$(18, 6)$ $D > 0$	$D = (-2)(-64) - (6)^2 = 12y - 36$
Saddle	max or min	$f_{xx} < 0, \text{ max}$

4. Student learning learning is dependent on two factors: Sleep (S , hours per night) and Caffeine (C , doses per day). The equation

$$L(S, C) = 3S^2 + 2C^2 + 4C$$

is bound by the constraint that $S^2 + C^2 = 40$. How much sleep and how much caffeine should a student get in order to maximize learning.

$$\begin{aligned} L &= 3S^2 + 2C^2 + 4C \quad \text{s.t.} \quad S^2 + C^2 = 40 \\ L_S &= 6S = 2\lambda S \quad \rightarrow \quad S = 0 \quad \text{or} \quad \lambda = 3 \\ L_C &= 4C + 4 = 2\lambda C \quad C = \sqrt{40} \\ \lambda = 3 & \quad 4C + 4 = 6C \quad \hookrightarrow L = 0 + 80 + 8\sqrt{10} \\ 2C &= 4 \quad S \geq S^2 + 4 = 40 \\ C &= 2 \quad S = 6 \\ L &= 108 + 8 + 8 = 124 \quad \boxed{\text{max}} \end{aligned}$$

5. We can define the *normal line* to a surface of the form $F(x, y, z) = k$ at the point (a, b, c) to be the line through (a, b, c) in the direction of ∇F .

(a) Find the normal line to the surface $xy + xz + yz = 14$ at the point $(2, 1, 4)$.

$$\vec{\nabla}F = \langle y+z, x+z, x+y \rangle \Big|_{(2,1,4)}$$

$$= \langle 5, 6, 3 \rangle$$

$$\vec{l}(t) = \langle 2, 1, 4 \rangle + t \langle 5, 6, 3 \rangle$$

(b) Prove that the normal line at any point on a sphere goes through the origin.

$$\text{Sphere: } x^2 + y^2 + z^2 = a^2$$

$$\vec{\nabla}F = \langle 2x, 2y, 2z \rangle \Big|_{(x_0, y_0, z_0)}$$

$$= \langle 2x_0, 2y_0, 2z_0 \rangle$$

$$\vec{l}(t) = \langle x_0, y_0, z_0 \rangle + t \langle 2x_0, 2y_0, 2z_0 \rangle$$

$$\text{when } t = -\frac{1}{2}$$

$$l = \langle 0, 0, 0 \rangle$$

6. A function $f(x, y)$ is harmonic if $f_{xx} + f_{yy} = 0$. Prove that $f(x, y) = e^x \sin(y)$ is harmonic.

$$f(x, y) = e^x \sin y$$

$$f_x = e^x \sin y$$

$$f_{xx} = e^x \sin y$$

$$f_{xx} + f_{yy}$$

$$= e^x \sin y + (-e^x \sin y) = 0.$$

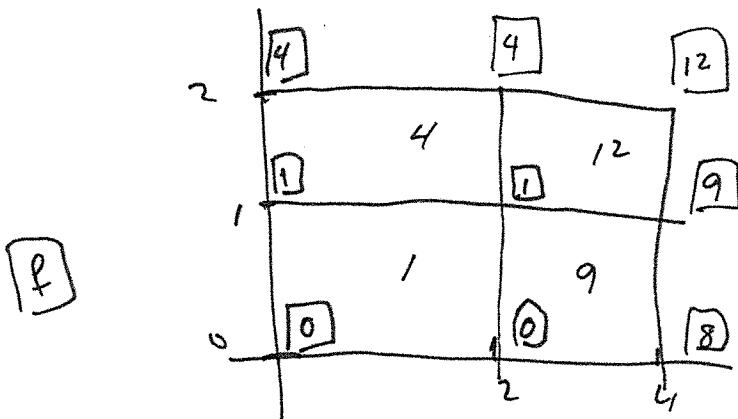
$$f_y = e^x \cos y$$

$$f_{yy} = e^x (-\sin y)$$

7. Give an overestimate of

$$\int_0^4 \int_0^2 x^2 - 2x + y^2 dy dx$$

using 4 subdivisions.



$$\text{Overest: } 1 \cdot 2 + 4 \cdot 2 + 9 \cdot 2 + 12 \cdot 2 \\ = 26 \cdot 2 = \underline{\underline{52}}$$

8. Determine

$$\iint_R y\sqrt{xy+1} dA$$

where $R = [0, 4] \times [0, 6]$

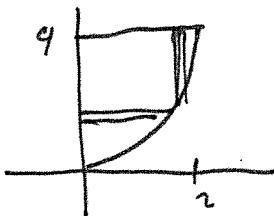
$$\begin{aligned} & \int_0^6 \int_0^4 y \sqrt{xy+1} dx dy \\ &= \int_0^6 \frac{2}{3} (xy+1)^{3/2} \Big|_0^4 dy \\ &= \frac{2}{3} \left[\int_0^6 (4y+1)^{3/2} - 1 dy \right] \end{aligned}$$

$$\begin{aligned} & \left. \frac{2}{3} \left[\frac{2}{5} \frac{(4y+1)^{5/2}}{4} - y \right] \right|_0^6 \\ &= \frac{2}{3} \left[\frac{2}{5} \frac{(25)}{4} - 6 \right] \\ &= \frac{2}{3} \left[\frac{2 \cdot 3125}{4} - 6 \right] \\ &= \left(\frac{3125}{3} - 4 \right) \end{aligned}$$

9. Consider

$$\iint_R x + y dA$$

where R the region in the first quadrant bound by $y = x^2$ and $y = 4$. Set this integral up using both orders of integration, and verify that your answers are the same.



$$\int_0^2 \int_{x^2}^4 x + y dy dx$$

$$= \int_0^2 xy + \frac{y^2}{2} \Big|_{x^2}^4 dx$$

$$= \int_0^2 (4x + 8) - \left(x^3 + \frac{x^4}{2}\right) dx$$

$$= 2x^2 + 8x - \left(\frac{x^4}{4} + \frac{x^5}{10}\right) \Big|_0^2$$

$$= 24 - 7.2 = \underline{\underline{16.8}}$$

$$\int_0^4 \int_0^{\sqrt{y}} x + y dx dy$$

$$= \int_0^4 \frac{x^2}{2} + xy \Big|_0^{\sqrt{y}} dy$$

$$= \int_0^4 \frac{y}{2} + y^{3/2} dy$$

$$= \frac{y^2}{4} + \frac{2}{5} y^{5/2} \Big|_0^4$$

$$= 4 + \frac{64}{5} = \underline{\underline{16.8}}$$