Math 22: Quiz the Sixth
October 22, 2021

KEY

You have the remainder of the period to complete this quiz. Please justify your answers where appropriate and READ ALL DIRECTIONS CAREFULLY. You may use a calculator for computation only.

1. Find the equation of the tangent plane to the surface

$$f(x, y) = x^2 + 3xy - y^3$$

at the point (1,3).

$$f(1,3) = 1 + 9 - 27 = -17$$

$$f_x = 2x + 3y \Big|_{(1,3)} = 11$$

$$f_y = 3x - 3y^2 \Big|_{(1,3)} = -24$$

$$z = -17 + 11(x-1) - 24(y-3)$$

2. Let

$$x^3y + y^3z + z^3x = 5$$

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, using either implicit differentiation or the Implicit Function Theorem.

$$3x^2y + y^3\frac{\partial z}{\partial x} + z^3 + 3z^2x\frac{\partial z}{\partial x} = 0$$

$$\rightarrow \quad \frac{\partial z}{\partial x} = \frac{-3x^2y - z^3}{y^3 + 3z^2x} = \frac{-F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = \frac{-(x^3 + 3y^2z)}{y^3 + 3z^2x}$$

(b) At which points can we *not* define $y = f(x, z)$? (An equation is fine here...)

$$F_y \neq 0 \quad \rightarrow \quad x^3 + 3y^2z \neq 0.$$

(c) Give the equation for the tangent plane at the point $(1, -1, 2)$

$$F_x = 3x^2y + z^3 = -3 + 8 = 5$$

$$F_y = 3y^2z + x^3 = 3 + 1 = 4$$

$$F_z = 3z^2x + y^3 = 12 - 1 = 11$$

$$5(x-1) + 4(y+1) + 11(z-2) = 0$$

3. Let $f(x, y) = \sqrt{x^2 + y^2}$, and let $x = s^2 - t^2$ and $y = 2st$. Calculate $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial x}{\partial s} = 2s \qquad \frac{\partial y}{\partial s} = 2t$$

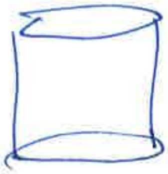
$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \qquad \frac{\partial x}{\partial t} = -2t \qquad \frac{\partial y}{\partial t} = 2s$$

$$\frac{\partial f}{\partial s} = \frac{x}{\sqrt{x^2+y^2}} \cdot 2s + \frac{y}{\sqrt{x^2+y^2}} \cdot 2t$$

$$\frac{\partial f}{\partial t} = \frac{x}{\sqrt{x^2+y^2}} \cdot (-2t) + \frac{y}{\sqrt{x^2+y^2}} (2s)$$

(For this week's **bonus**, pick several integer values for s and t, preferably ones that keep x positive, and comment on the values you get for x, y, and f.)

4. A scary Halloween decoration in the shape of a circular cylinder is oozing onto the ground. Its radius is increasing at a rate of 2 cm/min, while its height is decreasing at a rate of 3 cm/min. When the object has radius 4 cm and height 5 cm, is the object gaining or losing volume? Explain!



$V = \pi r^2 h$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= 2\pi r h\left(\frac{dr}{dt}\right) + \pi r^2\left(\frac{dh}{dt}\right) \longrightarrow 2\cdot\pi\cdot 4\cdot 5(2) + \pi(4)^2(-3)$$

$$80\pi - 48\pi = 32\pi \text{ cm}^3/\text{min}$$

Scary indeed!

5. Let $f(x,y) = x^2 + 3xy - y^3$ (as in exercise 1).

(a) Find the directional derivative of f at the point $(1,3)$ in the direction of the vector $\langle 3,4\rangle$

$\nabla f = \langle 11, -24\rangle$

$\vec{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$

$$D_{\vec{u}} f = \langle 11, -24\rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5}\rangle$$

$$= \frac{33}{5} - \frac{96}{5} = -\frac{63}{5}$$

(b) Find a direction for which the directional derivative of f is zero.

$\vec{u} \perp \nabla f$, so $\langle 24, 11\rangle$ works