KEY

Math 225: Exam the Second

You have two hours to complete this exam. You may use a calculator for computation only, and you should be prepared to show the relevant steps to a problem where necessary.

1. Consider the functions $f(x,y) = x^2 + y^2$, $g(x,y) = \sqrt{x^2+y^2}$, and $h(x,y) = \ln(x^2+y^2)$.

   (a) What familiar surfaces are the graphs of f and g?

   2 $f \to$ a paraboloid

   $g \to$ a cone

   (b) What familiar curve is a level curve for f, g, or h?

   For all 3, the curve is a circle

   2 $K = x^2+y^2$ $\quad K^2 = x^2+y^2$ $\quad e^K = x^2+y^2$

   (c) Plot the level curves for each of the three functions for $z = 1, 2, 3, 4$ and 5, with attention to spacing to show steepness.

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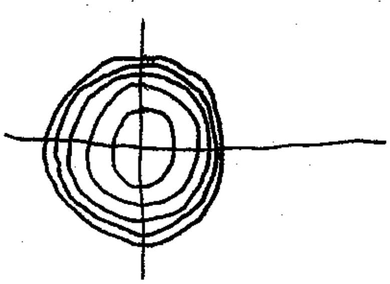
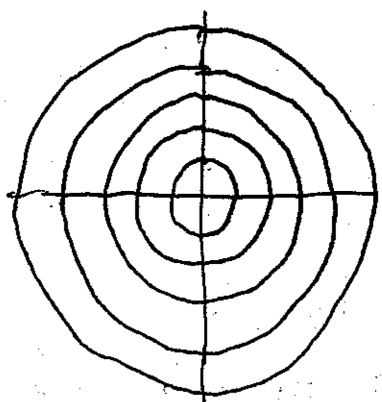
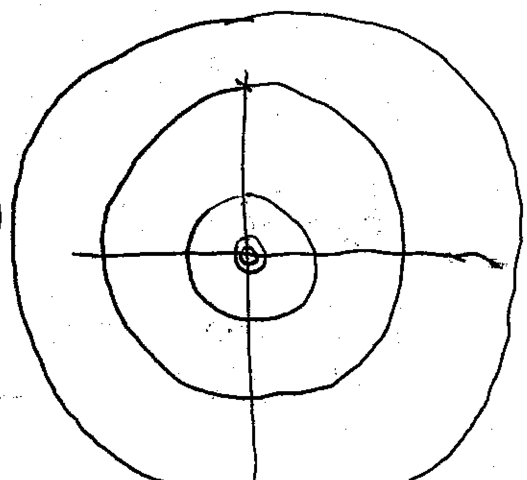


f



g



h

   (d) Which function is steepest? Which is least steep? Explain.

2 f is steepest
curves closer together

h is least steep
curves further apart

2. Find

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

if it exists.

If $x=0$ $\lim_{y\to 0} = \frac{(-y)^2}{y^2} = 1$

so limit DNE.

If $x=y$ $\lim_{y\to 0} \frac{0}{2y^2} = 0$

4

3. Suppose that for a continuous and differentiable function $f(x,y)$, we know that $f_x(x,y) = 2xy + x^3$. Give at least three possible functions that could be $f_y(x,y)$.

$f_x = 2xy + x^3$

$f_{xy} = 2x \longrightarrow f_{yx} = 2x$

$\longrightarrow f_y = x^2 + g(y)$

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So any of f_y x^2

$f_y = x^2 + \cos y$

$f_y = x^2 + \arcsin(e^y)$ work

4. Suppose that the temperature of a plate is given by the function $f(x, y) = \frac{y}{x^2}$.

(a) Suppose you are at the point $(2, 3)$. In what direction should you move so as to increase your temperature the most rapidly?

$f(x,y) = \frac{y}{x^2}$

$f_x = \frac{-2y}{x^3}$ @ $(2,3)$ $\to f_x = \frac{-3}{4}$

5 $f_y = \frac{1}{x^2}$ $\to f_y = \frac{1}{4}$

so, in the direction of $\langle \frac{-3}{4}, \frac{1}{4} \rangle$

(b) Suppose that you move from $(2, 3)$ towards $(5, -1)$. At what rate is your temperature changing?

$(2,3) \to (5,-1)$ has vector $\langle 3, -4 \rangle$

or $\vec{u} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

$\langle \frac{-3}{4}, \frac{1}{4} \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{-9}{20} + \frac{-4}{20} = -\frac{13}{20}$ units T / units d

5

(c) Approximate the temperature at $(2.04, 2.99)$.

$f(2,3) = \frac{3}{4}$

$z \approx z_0 + f_x(x - x_0) + f_y(y - y_0)$

5 $= \frac{3}{4} + (\frac{-3}{4})(2.04-2) + \frac{1}{4}(2.99-3)$

$= \frac{3}{4} + \frac{-.12}{4} + \frac{-.01}{4}$

~~3.1~~ $\boxed{\frac{2.87}{4}}$

5. Find and classify the critical points of $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - y^2$. (There are four of them.)

$f_x = x^2 - x = 0 \quad x = 0, 1$

$f_y = y^2 - 2y = 0 \quad y = 0, 2$

points
$(0,0) \to$ MAX
$(1,0) \to$ S.P.
$(0,2) \to$ S.P.
$(1,2) \to$ min

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$f_{xx} = 2x - 1$
$f_{yy} = 2y - 2$
$f_{xy} = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (2x-1)(2y-2)$

since $f_{xx} < 0$

$D(0,0) = (-1)(-2) > 0 \Rightarrow$ max or ~~min~~

$D(1,0) = (1)(-2) < 0 \to$ Saddlepoint

$D(0,2) = (-1)(2) < 0 \to$ Saddle point

$D(1,2) = (1)(2) > 0 \to$ ~~max~~ or min

since $f_{xx} > 0$

6. Find the following integrals. Where necessary or appropriate, you may reverse the order of integration or convert into another coordinate system.

(a)

$$\int_0^1 \int_0^1 y\cos(xy)\,dy\,dx$$

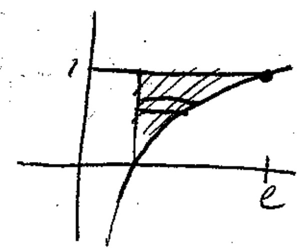
flip for ease $\int_0^1 \int_0^1 y\cos(xy)\,dx\,dy$

$$\cancel{= \int_0^1 \sin(xy)\,dy} = \int_0^1 \sin(xy)\Big|_0^1 dx = \int_0^1 \sin x\,dx$$

5 $= -\cos x \Big|_0^1 = \boxed{1-\cos 1}$

(b)

$$\int_1^e \int_{\ln x}^1 e^{e^y}\,dy\,dx$$



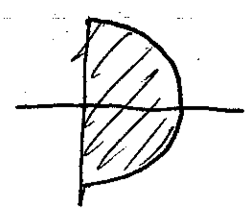
5 $\int_0^1 \int_1^{e^y} e^{e^y}\,dx\,dy$

$$\cancel{= \int_0^1 x e^{e^y}\Big|_1^{e^y} dy}$$

$$\cancel{= \int_0^1 e^y e^{e^y} - e^{e^y}}$$

(c)

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$$

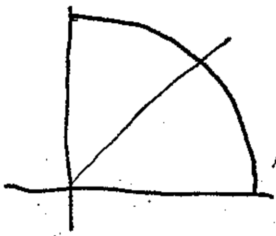


5 $\int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cdot r\,d\theta\,dr$

$$= \pi \int_0^2 r^3\,dr = \pi \frac{r^4}{4}\Big|_0^2 = 4\pi$$

7. Consider a plate in the shape of a quarter circle of radius 1 in the first quadrant. Suppose that the density of the plate is given by $\rho(x, y) = (x - y)^2$.

(a) Find the mass of the plate.



$$\rho(x,y) = (x-y)^2 = x^2+y^2-2xy$$

$$\text{Mass} = \iint \rho(x,y)\, dA$$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} (r^2 - 2r^2 \sin\theta\cos\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^3 - 2r^3 \sin\theta\cos\theta\, d\theta$$

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$$= \int_0^{\pi/2} \frac{r^4}{4} - \frac{r^4}{2}\sin\theta\cos\theta \Big]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2}\sin\theta\cos\theta\, d\theta$$

$$= \left[\frac{1}{4}\theta - \frac{1}{4}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{8} - \frac{1}{4} = (\pi - 2)/8$$

(b) Set up the integral to find $\overline{x}$. Convert this integral to polar coordinates, but DO NOT attempt to compute it.

$$\bar{x} = \frac{\iint x(x-y)^2\, dA}{\left(\frac{\pi-2}{8}\right)} = \frac{\int_0^1\int_0^{\pi/2} r^4\cos\theta - 2r^4\sin\theta\cos\theta\, d\theta}{\left(\frac{\pi-2}{8}\right)}$$

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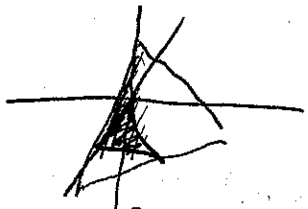
(c) Argue that, for this plate, $\overline{y} = \overline{x}$:

Both are true and need to be true

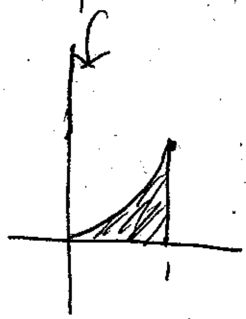
A) The plate is symmetric wrt the line $y=x$

3 B) The density is symmetric w.r.t the line $y=x$

8. Find the volume of the region bounded by the planes $y = 0, x = 1, z = 0, y = x^2$ and $4x + 2y + z = 7$



$$\int_0^1 \int_0^{x^2} 7 - 4x - 2y \, dy \, dx$$



$$= \int_0^1 7y - 4xy - y^2 \Big|_0^{x^2} dx$$

$$= \int_0^1 7x^2 - 4x^3 - x^4 \, dx$$

$$= \left[\frac{7}{3}x^3 - x^4 - \frac{x^5}{5} \right]_0^1$$

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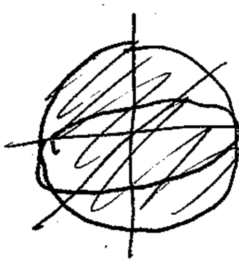
$$= \frac{7}{3} - 1 - \frac{1}{5} = \frac{4}{3} - \frac{1}{5} = \frac{17}{15}$$

9. Find, using Calculus, the volume of a sphere of radius R.



$0 \le \rho \le R$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$\text{Vol} = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta$$

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$$= \int_0^{2\pi} \left[\frac{R^3}{3}(-\cos\phi) \right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{R^3}{3}(1+1) \, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} R^3 d\theta = \boxed{\frac{4\pi R}{3}}$$

10. In a homework exercise, we showed that

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

has finite volume over the unit disk centered at the origin, even though the function grows unbounded near the origin. Does this function have a finite surface area over the unit disk? Set up and simplify an appropriate integral enough to give an explanation as to why or why not.

$z = (x^2+y^2)^{-1/2}$

$f_x = -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2x = \frac{-x}{\sqrt{(x^2+y^2)^3}}$

$f_y = -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2y = \frac{-y}{\sqrt{(x^2+y^2)^3}}$

$$S.A. = \iint_R \sqrt{f_x^2+f_y^2+1}\, dA$$

$$= \iint \sqrt{\frac{x^2}{(x^2+y^2)^3} + \frac{y^2}{(x^2+y^2)^3} + 1}\, dA$$

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$$= \iint \sqrt{\frac{x^2+y^2}{(x^2+y^2)^3} + 1}\, dA$$

/ polars

$$= \int_0^{2\pi}\int_0^1 \sqrt{\frac{\cancel{r^2}}{(r^2)^{\cancel{3}\,2}} + 1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{r^4} + 1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{\sqrt{r^4+1}}{r^2}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{\sqrt{r^4+1}}{r}\; dr\,d\theta$$

and since the relative power on r is greater than 1, we expect the integral to converge!

So S.A. is finite.

11. (Extra Credit) Which mathematical term comes...

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(a) from the Latin word for 'snail'? Limaçon

(b) from the Greek word for 'heart'? cardioid

(c) from the Greek words for 'alongside' and 'throw'? parabola.