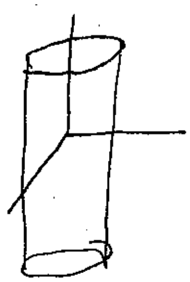
Key

Math 225: Exam the First

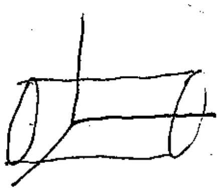
You have two hours to complete this exam. You may use a calculator for computation only, and you should be prepared to show the relevant steps to a problem where necessary.

1. (a) Give the equation, in rectangular coordinates, of the cylinder of radius 6 centered around the z-axis.



$x^2+y^2=36$

(b) Give the equation, in rectangular coordinates, of the cylinder of radius 6 centered around the y-axis.



$x^2+z^2=36$

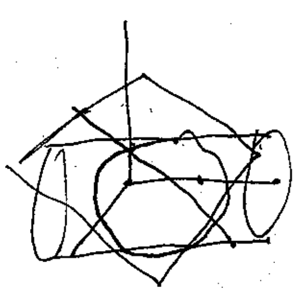
(c) Find, in parametric form, the equation of the curve of intersection of the cylinder in part (b) with the plane $y+4z=3$, and describe the curve.

$x^2+z^2=36$
$y+4z=3$

$z=t$
$x=\pm\sqrt{36-t^2}$
$y=3-4t$

OR

$x=6\cos t$
$z=6\sin t$
$y=3-24\sin t$



The curve is an ellipse

2. Let ℓ_1 be the line through the two points (-3, 1, 0) and (1,1,2), and ℓ_2 be the line through the points (6, 2, 6) and (3, -1, 0).

(a) Find the point of intersection of ℓ_1 and ℓ_2.

ℓ_1 point $(-3,1,0)$

vector $\langle 4,0,2\rangle$

$x = -3+4t$

$y = 1$

$z = 2t$

ℓ_2 point $(6,2,6)$

vector $\langle -3,-3,-6\rangle$

(write in s)

$x = 6-3s$

$y = 2-3s$

$z = 6-6s$

$2-3s=1$

$s=\frac{1}{3}$

$2t = 6-6(\frac{1}{3})$

$= 6-2$

$t=2$

$-3+4(2)=5$

1

$2(2)=4$

$6-3(\frac{1}{3})=5$

1

$6-6(\frac{1}{3})=4$

point $(5,1,4)$

(b) Find the plane that contains both lines.

$\vec{n} = \langle 4,0,2\rangle \times \langle -3,-3,-6\rangle$

$\langle 6, 18, -12\rangle$ point: $(5,1,4)$ (or any other)

plane: $6(x-5)+18(y-1)-12(z-4)=0$.

3. (a) The equation in spherical coordinates:

$$\rho = 4 \sin\phi(\cos\theta + \sin\theta)$$

defines a sphere. Find its center and radius. (Hint: Multiply both sides by ρ and convert to rectangular coordinates).

$$\rho^2 = 4\rho \sin\phi (\cos\theta + \sin\theta)$$

$$x^2 + y^2 + z^2 = 4x + 4y$$

$$\underset{+4}{x^2 - 4x} + \underset{+4}{y^2 - 4y} + z^2 = \underset{+8}{0}$$

$$(x-2)^2 + (y-2)^2 + z^2 = 8$$

center: $(2, 2, 0)$

radius: $\sqrt{8}$.

(b) Convert the center point to cylindrical and spherical coordinates.

rect: $(2, 2, 0)$

$\downarrow$

$r = \sqrt{x^2+y^2} = \sqrt{8}$

$\theta = \arctan\left(\frac{y}{x}\right) = \arctan(1) = \frac{\pi}{4}$

$z = z = 0$

cyl: $(\sqrt{8}, \frac{\pi}{4}, 0)$

rect: $2, 2, 0$

$\downarrow$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{8}$

$\theta = \frac{\pi}{4}$ (from before)

$\phi = \arccos\left(\frac{z}{\rho}\right)$

$= \arccos(0) = \pi/2$

sph: $(\sqrt{8}, \frac{\pi}{4}, \frac{\pi}{2})$

4. (a) Find the equation of the tangent line to the curve $\mathbf{r}(t) = \langle 4-t, 3t-t^2, t\rangle$ at the point when $t = 0$.

$\vec{r}(0) = (4,0,0)$ $\quad r'(t) = \langle -1, 3-2t, 1\rangle$

$\vec{r}'(0) = \langle -1, 3, 1\rangle$

Tangent Line: $\langle 4,0,0\rangle + t\langle -1, 3, 1\rangle$

(b) Using your work in part (a), find $\mathbf{T}(0)$. (Do NOT try to calculate a generic formula for $\mathbf{T}(t)$).

$T(0) = \dfrac{r'(0)}{|r'(0)|}$, so convert $\langle -1,3,1\rangle$ to a unit vector

$T(0) = \langle \frac{-1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\rangle$

(c) We can show that $\mathbf{T}'(0) = \langle -3, -2, 3\rangle$. Find $\mathbf{N}(0)$, $\mathbf{B}(0)$, and the osculating plane to the curve at $t = 0$.

$N(0) = \dfrac{T'(0)}{|T'(0)|} = \left\langle \frac{-3}{\sqrt{22}}, \frac{-2}{\sqrt{22}}, \frac{3}{\sqrt{22}}\right\rangle$

$B = T \times N = $

$\langle \frac{-1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\rangle$

$\times \langle \frac{-3}{\sqrt{22}}, \frac{-2}{\sqrt{22}}, \frac{3}{\sqrt{22}}\rangle$

$\left\langle \frac{9+2}{\sqrt{11}\cdot\sqrt{22}}, \frac{-3+3}{\sqrt{11}\times\sqrt{22}}, \frac{2+9}{\sqrt{11}\times\sqrt{22}}\right\rangle$

$= \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\rangle$

osc. plane: $\frac{1}{\sqrt{2}}(x-4) + 0(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0.$

5. Show that if vectors $\mathbf{x}-\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$ are orthogonal, then $\mathbf{x}$ and $\mathbf{y}$ must have the same length.

$(\vec{x}-\vec{y}) \perp (\vec{x}+\vec{y})$

$(\vec{x}-\vec{y})\cdot(\vec{x}+\vec{y})=0$

$\Rightarrow \vec{x}\cdot\vec{x} - \cancel{\vec{y}\cdot\vec{x} + \vec{x}\cdot\vec{y}} - \vec{y}\cdot\vec{y} = 0$

$\Rightarrow |\vec{x}|^2 - |\vec{y}|^2 = 0$

$|\vec{x}|^2 = |\vec{y}|^2$

$|\vec{x}| = |\vec{y}|$

6. Let $\mathbf{x}$ and $\mathbf{y}$ be unit vectors. What are the minimum and the maximum magnitude of $\mathbf{x}\times\mathbf{y}$, and what is the geometric relationship between $\mathbf{x}$ and $\mathbf{y}$ when these are achieved? Why?

$|\vec{x}\times\vec{y}| = |\vec{x}|\cdot|\vec{y}|\cdot \sin\theta$

$= 1\cdot 1\cdot \sin\theta$

so

$0 \leq |\vec{x}\times\vec{y}| \leq 1$

$|\vec{x}\times\vec{y}| = 0$ when $\theta = 0$, ie, when $\vec{x}$ & $\vec{y}$ are parallel

$|\vec{x}+\vec{y}| = 1$ when $\theta = \frac{\pi}{2}$, ie, when $\vec{x}$ & $\vec{y}$ are perpendicular

7. Suppose that a particle is moving with acceleration

$$\mathbf{a}(t) = \langle 6t, \cos(t), e^t \rangle$$

and that the object starts with initial velocity vector $\langle 2, 1, 2 \rangle$ and initial position vector $\langle 0, 1, 3 \rangle$. Find the position of the object when $t = 1$.

$\vec{a}(t) = \langle 6t, \cos t, e^t \rangle$

$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$

$= \langle 2, 1, 2 \rangle + \langle 3t^2, \sin t, e^t \rangle - \langle 0, 0, 1 \rangle$

$= \langle 3t^2 + 2, \sin t + 1, e^t + 1 \rangle$

$s(t) = \vec{s}(0) + \int_0^t \vec{v}(u)\,du$

$= \langle 0, 1, 3 \rangle + \langle t^3 + 2t, -\cos t + t, e^t + t \rangle - \langle 0, -1, 1 \rangle$

$= \langle t^3 + 2t, -\cos t + t + 2, e^t + t + 2 \rangle$

$s(1) = \langle 3, 3 - \cos 1, 3 + e \rangle$

8. (Extra Credit) Prove that a straight line has zero curvature.

Line: $\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

$\vec{r}\,'(t) = \langle a, b, c \rangle$

$\vec{r}\,''(t) = \langle 0, 0, 0 \rangle$

Thus $|r' \times r''| = 0$ so $K = \dfrac{|r' \times r''|}{|r'|^3} = 0$.