

## SAMPLE EXAM 3 SOLUTIONS

1. Compute:  $\iint_R x e^{xy} dA$ ,  $R = [0, 1] \times [0, 1]$ .

$$\int_0^1 \int_0^1 x e^{xy} dy dx = \int_0^1 e^{xy} \Big|_0^1 dx = \int_0^1 e^x - 1 dx = e^x - x \Big|_0^1 = e - 1 - 1 = e - 2$$

2. Compute:  $\iint_R 1/x dA$ ,  $R = \{(x, y) \mid 1 \leq y \leq e, y^2 \leq x \leq y^4\}$ .

$$\begin{aligned} \int_1^e \int_{y^2}^{y^4} \frac{1}{x} dx dy &= \int_1^e \ln y^4 - \ln y^2 dy = \int_1^e 4 \ln y - 2 \ln y dy = \int_1^e 2 \ln y dy \\ &= 2[y \ln y - y]_1^e = 2(e - e - 0 + 1) = 2 \end{aligned}$$

3. Compute:  $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx dy$ .

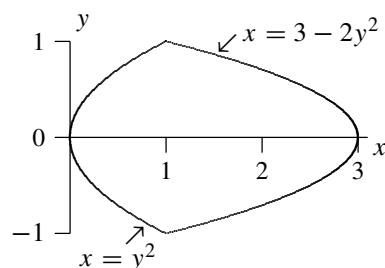
$$\begin{aligned} \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} dx dy &= \int_0^1 \int_0^{x^2} \sqrt{x^3 + 1} dy dx = \int_0^1 x^2 \sqrt{x^3 + 1} dx \\ &= \frac{1}{3} \frac{2}{3} (x^3 + 1)^{3/2} \Big|_0^1 = \frac{2}{9} 2^{3/2} - \frac{2}{9} \end{aligned}$$

4. Find the volume under  $z = x^2 + y^2$  and above  $[-2, 2] \times [-3, 3]$ .

$$\int_{-2}^2 \int_{-3}^3 x^2 + y^2 dy dx = \int_{-2}^2 x^2 y + \frac{y^3}{3} \Big|_{-3}^3 dx = \int_{-2}^2 6x^2 + 18 dx = \frac{6x^3}{3} + 18x \Big|_{-2}^2 = 104$$

5. Set up an integral for the surface area of  $z = y^2 - x + 3$  above the region shown. Do not evaluate the integral.

$$\int_{-1}^1 \int_{y^2}^{3-2y^2} \sqrt{2 + 4y^2} dx dy =$$



6. Find the volume under  $z = xy$  and above the region inside  $r = 1 + \cos \theta$  in the first quadrant.

$$\int_0^{\pi/2} \int_0^{1+\cos\theta} \sin \theta \cos \theta r^3 dr d\theta = \int_0^{\pi/2} \frac{(1 + \cos \theta)^4}{4} \sin \theta \cos \theta d\theta$$

Now use  $u = 1 + \cos \theta$ :

$$\int_1^2 \frac{u^4}{4} (u - 1) du = \frac{1}{4} \left( \frac{2^6 - 1}{6} - \frac{2^5 - 1}{5} \right) = \frac{43}{40}$$

7. A flat plate has the shape bounded by the parabola  $y = 9 - x^2$  and the  $x$ -axis; the density is given by  $\rho(x, y) = x^2 y$ . Find the total mass and the  $y$  coordinate of the center of mass.

$$M = \int_{-3}^3 \int_0^{9-x^2} x^2 y dy dx = \frac{1}{2} \int_{-3}^3 x^2 (9 - x^2)^2 dx = \frac{1}{2} \int_{-3}^3 81x^2 - 18x^4 + x^6 dx =$$

$$3^6 - \frac{18 \cdot 3^5}{5} + \frac{3^7}{7} = \frac{5832}{35}$$

$$\bar{y} = \frac{1}{M} \int_{-3}^3 \int_0^{9-x^2} x^2 y^2 dy dx = \frac{1}{3M} \int_{-3}^3 x^2 (9 - x^2)^3 dx =$$

$$\frac{1}{3M} \int_{-3}^3 3^6 x^2 - 3^5 x^4 + 3^3 x^6 - x^8 dx = \frac{2}{3M} \left( 3^8 - \frac{3^{10}}{5} + \frac{3^{10}}{7} - 3^7 \right) = \frac{23328}{35M} = 4$$

8. Compute:  $\int_0^2 \int_0^{\sqrt{9-x^2}} \int_0^{x^2} yz dy dz dx$ .

$$\int_0^2 \int_0^{\sqrt{9-x^2}} \int_0^{x^2} yz dy dz dx = \frac{1}{2} \int_0^2 \int_0^{\sqrt{9-x^2}} x^4 z dz dx = \frac{1}{4} \int_0^2 x^4 (9 - x^2) dx =$$

$$\frac{9 \cdot 2^3}{5} - \frac{2^5}{7} = \frac{344}{35}$$

9. Compute:  $\iiint_R x^3 + xy^2 dV$ , where  $R$  is the three dimensional region in the first octant that is under  $z = 1 - x^2 - y^2$ .

$$\begin{aligned} \int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} \cos \theta r^4 dz dr d\theta &= \int_0^{\pi/2} \int_0^1 \cos \theta r^4 (1 - r^2) dr d\theta = \\ \int_0^{\pi/2} \cos \theta \left[ \frac{r^5}{5} - \frac{r^7}{7} \right]_0^1 d\theta &= \frac{2}{35} \sin \theta \Big|_0^{\pi/2} = \frac{2}{35} \end{aligned}$$

10. Find the mass of a hemisphere of radius 1 if the density is  $\sigma(x, y, z) = z$ , assuming that the sphere is centered at the origin and the hemisphere is the upper half.

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho \cos \phi \rho^2 \sin \phi d\rho d\phi d\theta &= \frac{1}{4} \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi d\phi d\theta = \\ \frac{1}{4} \int_0^{2\pi} \frac{1}{2} [\sin^2 \phi]_0^{\pi/2} d\theta &= \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \end{aligned}$$