KEY

Math 225: Exam the First
February 24, 2012

You have 90 minutes to complete this exam. You may use a calculator for arithmetic only, and be sure to give algebraic justification to your answers where necessary.

1. Let the points $A = (2, 1, 5)$, $B = (-1, 3, 4)$ and $C = (3, 0, 6)$ form a triangle.

   (a) Use the distance formula (thrice!) to find the perimeter of the triangle. (Your answer will be a sum of three radicals).

$|AB| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$
$|AC| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$
$|BC| = \sqrt{4^2+3^2+2^2} = \sqrt{29}$

   (b) Is the triangle right, acute, or obtuse? Explain.

$\vec{AB}$ $\langle -3, 2, -1\rangle$
$\vec{AC}$ $\langle 1, -1, 1\rangle$

$\vec{AB}\cdot\vec{AC} = (-3 -2 -1) = -6 < 0$

So $\measuredangle A$, and hence $\triangle ABC$ is OBTUSE

   (c) Find the area of the triangle. (Hint: It is half of the area of a certain parallelogram.)

$\vec{AB} = \langle -3, 2, -1\rangle$
$\times\ \vec{AC} = \langle 1, -1, 1\rangle$

$\langle 1, 2, 1\rangle$ So $\text{Area}_{\text{pgram}} = |\langle 1,2,1\rangle| = \sqrt{6}$

So $\text{Area}_{\text{triangle}} = \sqrt{6}/2$

   (d) Find the equation of the plane containing the triangle.

Plane: $\vec{n} = \langle 1, 2, 1\rangle$
Point $= (2, 1, 5)$
eqn: $1(x-2) + 2(y-1) + 1(z-5) = 0$
or $x + 2y + z = 9$

2. (a) Find two unit vectors orthogonal to $\langle 1,1,1\rangle$ and $\langle 2,0,1\rangle$.

$$\begin{array}{r} \langle 1,1,1\rangle \\ \times\ \langle 2,0,1\rangle \\ \hline \langle 1,1,-2\rangle \end{array}$$

$\nearrow \frac{1}{\sqrt{6}}\langle 1,1,-2\rangle$

$\searrow \frac{1}{\sqrt{6}}\langle -1,-1,2\rangle$

(b) Prove that for any vectors $\mathbf{a}$ and $\mathbf{b}$, that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b})$.

$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$$

$$= \underbrace{\vec{a}\times\vec{a}}_{\to 0} - \vec{b}\times\vec{a} + \vec{a}\times\vec{b} - \underbrace{\vec{b}\times\vec{b}}_{\to 0}$$

$$= \underbrace{(-\vec{b}\times\vec{a})}_{\vec{a}\times\vec{b}} + (\vec{a}\times\vec{b})$$

$$\vec{a}\times\vec{b} + \vec{a}\times\vec{b} = 2(\vec{a}\times\vec{b})$$

3. Let $\mathbf{a} = \langle 2,3,-1\rangle$, $\mathbf{b} = \langle 4,-2,2\rangle$, and $\mathbf{c} = \langle -4,-6,2\rangle$

(a) Find $\mathbf{proj_a b}$ and $\mathbf{proj_a c}$.

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a}$$

$$= \frac{8-6-2}{14}\vec{a} = 0\vec{a} = \vec{0}$$

$$\text{proj}_{\vec{a}}\vec{c} = \frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{a}}\vec{a}$$

$$= \frac{-8-18-2}{14}\vec{a}$$

$$= -\frac{28}{14}\vec{a} = -2\vec{a} = \langle -4,-6,2\rangle$$

$$= \vec{c}$$

(b) Comment on your answers to part (a) in light of the geometric relationships between $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

$\vec{a}\perp\vec{b}$, thus $\text{proj}_{\vec{a}}\vec{b} = \vec{0}$

$\vec{a} \parallel \vec{c}$, thus $\text{proj}_{\vec{a}}\vec{c} = \vec{c}$.

4. Consider the vector valued function $\mathbf{r}(t) = \langle \cos(t), -\cos(t), \sin(t) \rangle$. This function is the curve of intersection of which cylinder and which plane?

Cyl: $x^2 + z^2 = 1$

plane $x + y = 0$.

5. Consider the elliptical helix given by the vector valued function $\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t), t \rangle$.

(a) Find the tangent line to this curve when $t = \frac{\pi}{2}$

$\vec{r}(\frac{\pi}{2}) = \langle 0, 3, \frac{\pi}{2} \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 3\cos t, 1 \rangle$

$\vec{r}\,'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

$\vec{l}(t) = \langle 0, 3, \frac{\pi}{2} \rangle + t\langle -2, 0, 1 \rangle$

(b) Set up, but do not compute, the integral which gives the length of this curve from $t = 0$ to $t = 2\pi$.

$$\int_0^{2\pi} \sqrt{4\sin^2 t + 9\cos^2 t + 1}\, dt$$

$$\uparrow$$

$(x')^2 \quad (y')^2 \quad (z')^2$

(c) Find the curvature of $\mathbf{r}(t)$ when $t = \pi$.

$r'(t) = \langle -2\sin t, 3\cos t, 1 \rangle \xrightarrow{\pi} \langle 0, -3, 1 \rangle$

$r''(t) = \langle -2\cos t, -3\sin t, 0 \rangle \xrightarrow{\pi} \langle 2, 0, 0 \rangle$

$\overline{r'(t) \times r''(t)} = \langle 0, 2, 6 \rangle$

$k = \frac{\sqrt{40}}{(\sqrt{10})^3} = \frac{2\sqrt{10}}{(\sqrt{10})^3} = \frac{1}{5}$

6. (a) What is the domain of $f(x,y) = \dfrac{1}{x^2+y^2+1}$?

x, y all real #'s

(no restrictions, as $x^2+y^2+1 > 0$.

(b) For which values k can we find level curves of the form $z = k$ when $z = \dfrac{1}{x^2+y^2+1}$?

$\infty > x^2+y^2+1 \geq 1$

so $0 < \dfrac{1}{x^2+y^2+1} \leq 1$ So $0 < k \leq 1$

(c) What shape do these level curves take? Draw some of them.

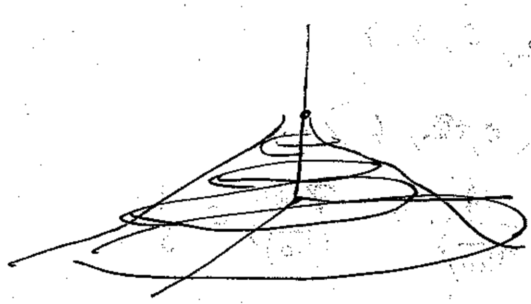
$k = \dfrac{1}{x^2+y^2+1} \Rightarrow x^2+y^2+1 = \dfrac{1}{k}$

$x^2+y^2 = \dfrac{1}{k} - 1$

Circles of radius $\frac{1}{k} - 1$ $r \to \infty$ as $k \to 0$

~~(d) What happens to the curves as k gets arbitrarily large?~~ and $\uparrow$

(e) Draw a rough sketch of $z = \frac{1}{x^2+y^2+1}$

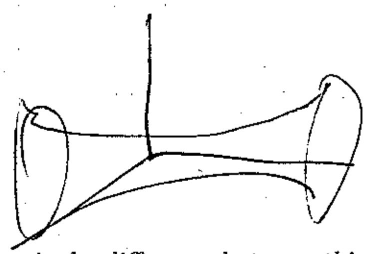


7. (a) Which surface is given by the equation $4x^2 - 16y^2 + z^2 = 16$?

$4x^2 + z^2 = 16 + 16y^2$

hyperboloid of one sheet

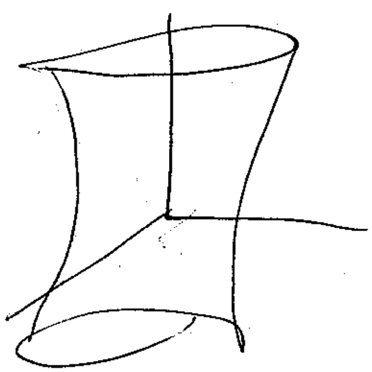
about y-axis



(b) What is the difference between this surface and the surface given by $4x^2 + 16y^2 - z^2 = 16$?

Both are hyperboloids of one sheet.

One is centered around y axis, this is around the z-axis



Bonus: Terrible Math Puns. Fill in the blank with the appropriate adverb.

1. "I can only touch base with you at one point later today," Tom said tangentially.
2. "Graphing $x^2 - y^2 = 1$ is the hardest thing I've ever done in my whole life," Tom said hyperbolically.
3. "The vector is perpendicular to the plane. So what?" Tom said normally.