KEY

Math 225: Exam the Second
April 13, 2012

You have 90 minutes to complete this exam. You may use a calculator for arithmetic only, and be sure to give algebraic justification to your answers where necessary.

1. (a) Find the tangent plane to $f(x,y) = x^3 - 3xy^2$ at the point $(3,4)$.

$f(3,4) = 3^3 - 3\cdot 3\cdot 4^2 = 9(3-16) = -117$

$f_x(3,4) = 3x^2 - 3y^2\big|_{(3,4)} = 3(9-16) = -21$

$f_y(3,4) = -6xy\big|_{(3,4)} = -72$

T plane: $z = -117 - 21(x-3) - 72(y-4)$

(b) Find $D_{\mathbf{u}}f$ as you move from (3,4) towards the origin.

$(3,4) \to (0,0)$

$\langle -3,-4\rangle \xrightarrow{\vec{u}} \langle \frac{-3}{5}, \frac{-4}{5}\rangle$

$D_{\vec{u}}(f) = \vec{\nabla} f \cdot \vec{u}$

$= \langle -21, -72\rangle \cdot \langle \frac{-3}{5}, \frac{-4}{5}\rangle$

$= \left(\frac{63}{5} + \frac{288}{5}\right) = \frac{351}{6}$

(c) A *harmonic function* $f(x,y)$ is one that satisfies $f_{xx} + f_{yy} = 0$. Prove that $f(x,y) = x^3 - 3xy^2$ is a harmonic function.

$f_x = 3x^2 - 3y^2 \qquad f_{xx} = 6x$

$f_y = -6xy \qquad + f_{yy} = -6x$

$6x - 6x = 0$ ✓

2. Find a function $f(x, y)$ such that $f_x = x^2 + 2xy + y^2$ and $f_y = x^2 + 2xy + y^2$. Also: Give another (hopefully interesting) function such that $f_x = f_y$.

$$f_x = x^2 + 2xy + y^2$$

$$f = \frac{x^3}{3} + x^2y + xy^2 + g(y)$$

$$f_y = 0 + x^2 + 2xy + g'(y) = x^2 + 2xy + y^2$$

$$g'(y) = y^2, \quad g(y) = \frac{y^3}{3}$$

$$f(x,y) = \frac{x^3}{3} + x^2y + xy^2 + \frac{y^3}{3}$$

Note: $f(x,y) = \frac{1}{3}(x+y)^3$, which makes sense since $f_x = f_y = (x+y)^2$

other possibilities $f(x,y) = (x+y)^5 \rightarrow f_x = f_y = 5(x+y)^4$, etc.

3. Let $x^2 + xy - y^2 + z^2 = 6$. Find formulas for $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$, and give an algebraic description of where these partials are defined.

IFT : $F(x,y,z) = x^2 + xy - y^2 + z^2 - 6 = 0$

$$\frac{\partial y}{\partial x} = \frac{-F_x}{F_y} = \frac{2x+y}{1-2y}$$

both are defined when $1 - 2y \neq 0$

or $y = \frac{1}{2}$

$$\frac{\partial y}{\partial z} = \frac{-F_z}{F_y} = \frac{2z}{1-2y}$$

4. Find and classify the critical points of $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - 4x - y$. (There are four of them).

$f_x = x^2 - 4 = 0 \qquad x = \pm 2$

$f_y = y^2 - 1 = 0 \qquad y = \pm 1$

CP's $(2,1) \qquad (2,-1)$

$(-2,1) \qquad (-2,-1)$

test: $D = (2x)(2y) - 0 = 4xy$

$D(2,1) > 0$ max or min ← since $f_{xx} \geq 0$

$D(-2,1) < 0$ saddle

$D(2,-1) < 0$ saddle

$D(-2,-1) > 0$ max a min since $f_{xx} < 0$

5. Find the volume below $z = xe^{xy}$ and above the box $[1,2] \times [1,3]$

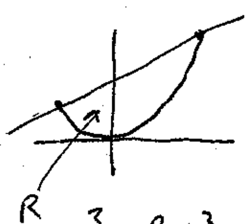
$$\iint_R xe^{xy}\,dA = \int_1^2\int_1^3 xe^{xy}\,dy\,dx$$

$$= \int_1^2 e^{xy}\Big|_1^3\,dx$$

$$= \int_1^2 e^{3x} - e^x\,dx$$

$$= \frac{e^{3x}}{3} - e^x\Big|_1^2 = \left(\frac{e^6}{3} - e^2\right) - \left(\frac{e^3}{3} - e\right)$$

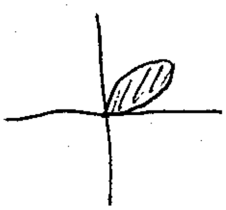
6. Find $\iint_R xy \, dA$ where R is bound by $y = x^2$ and $y = 2x + 3$.



$$x^2 = 2x + 3$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = 3, -1$$

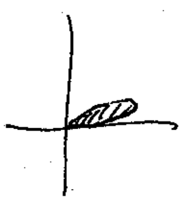
$$\int_{-1}^{3} \int_{x^2}^{2x+3} xy \, dy \, dx = \int_{-1}^{3} \left. \frac{xy^2}{2} \right|_{x^2}^{2x+3} dx$$
$$= \int_{-1}^{3} \frac{x(2x+3)^2}{2} - \frac{x^5}{2} \, dx$$
$$= \frac{1}{2} \int_{-1}^{3} 4x^5 + 12x^3 + 9x - x^5 \, dx = \frac{1}{2} \left[\frac{x^6}{2} + 3x^4 + \frac{9}{2} x^2 \right]_{-1}^{3} = \underline{640}$$

7. (a) Find the area inside one leaf of the four-leafed rose $r = \sin(2\theta)$.



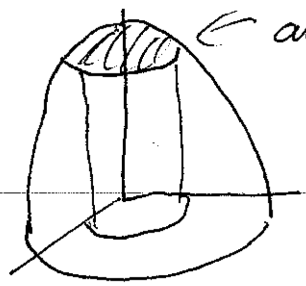
$$\int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} 1 \, \underline{r \, dr \, d\theta}$$
$$= \frac{1}{2} \int_0^{\pi/2} (\sin 2\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta$$
$$= \frac{1}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{8}$$

(b) Find the area inside one leaf of the eight-leafed rose $r = \sin(4\theta)$.



$$\int_0^{\pi/4} \int_0^{\sin 4\theta} 1 \, r \, dr \, d\theta$$
$$= \frac{1}{2} \int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{4} \int_0^{\pi/4} 1 - \cos 8\theta \, d\theta$$
$$= \frac{1}{4} \left[\theta - \sin 8\theta \right] d\theta = \frac{\pi}{16}$$

8. Find the surface area of the upper hemisphere of $x^2 + y^2 + z^2 = a^2$ inside the cylinder $x^2 + y^2 = b^2$. (it is assumed that $a > b$).



$z = \sqrt{1-x^2-y^2} \Rightarrow f_x = \frac{-x}{\sqrt{a^2-x^2-y^2}}$

$\Rightarrow f_y = \frac{-y}{\sqrt{a^2-x^2-y^2}}$

$$\iint_R \sqrt{1+f_x^2+f_y^2}\, dA$$

$$\iint_R \sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\, dA$$

$$= \iint_R \sqrt{\frac{a^2}{a^2-x^2-y^2}}\, dA$$

$$= \iint_R \frac{a}{\sqrt{a^2-(x^2+y^2)}}\, dA = a\int_0^{2\pi}\int_0^b \frac{r}{\sqrt{a^2-r^2}}\, dr\, d\theta$$

$$= 2a\int_0^{2\pi} -(a^2-r^2)^{1/2}\Big|_0^b\, d\theta$$

$$= -2a\int_0^{2\pi}\left[-\sqrt{a^2-b^2}+a\right] d\theta$$

$$= 4\pi a(a-\sqrt{a^2-b^2})$$

Bonus: If you went to an interesting talk (or poster) at the Undergraduate Conference, tell me about it.