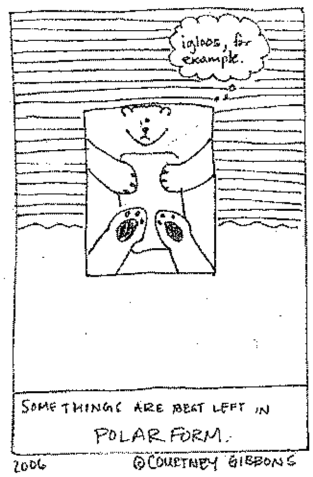
KEY

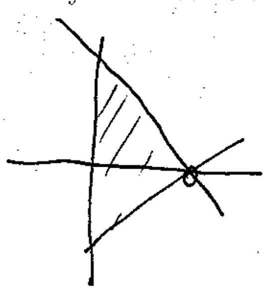
Math 225: Quiz the Ninth

April 5, 2012

This quiz is closed book and closed notes. Please justify all of your answers. You have 40 minutes.



1. Find the center of mass of the plate bound by the lines $y = x - 1$, $y = -x + 1$ and $x = 0$, where the plate has density $\rho(x, y) = x$. (You may use symmetry where appropriate to save yourself some work).



$$\text{mass} = \int_0^1 \int_{x-1}^{-x+1} x \, dy \, dx$$

$$= \int_0^1 x(-2x+2)\,dx = \int_0^1 -2x^2 + 2x \, dx$$

$$= -\frac{2}{3}x^3 + x^2 \Big|_0^1 = 1/3$$

$$\bar{x} = \int_0^1 \int_{x-1}^{-x+1} x^2 \, dy\, dx = \int_0^1 x^2(-2x+2)\,dx = \int_0^1 -2x^3 + 2x^2 \, dx$$

$$= -\frac{1}{2}x^4 + \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

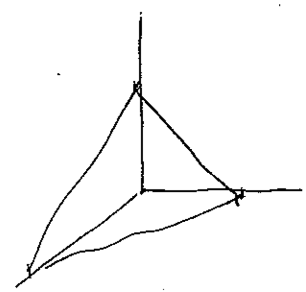
$$\bar{x} = \frac{1/6}{1/3} = 1/2.$$

$\bar{y} = 0$ by symmetry of both the plate & the density

center of mass $= (\frac{1}{2}, 0)$

2. Use a triple integral to find the volume in the first octant bound by the plane $x+3y+5z=15$.

$$\iiint_E 1\, dV = \int_0^3 \int_0^{5-\frac{5}{3}z} \int_0^{15-3y-5z} 1\, dx\, dy\, dz$$



$$= \int_0^3 \int_0^{5-\frac{5}{3}z} 15-3y-5z\, dy\, dz$$

$0 \le x \le 15-3y-5z$

$0 \le y \le 5-\frac{5}{3}z$

$0 \le z \le 3$

$$= \int_0^3 15y - \frac{3}{2}y^2 - 5yz \Big|_0^{5-5/3 z} dz$$

$$= \int_0^3 \underbrace{15(5-5/3 z)}_{75-75/3 z} - \frac{3}{2}\left(5-\frac{5}{3}z\right)^2 - 5(5-5/3 z)z\, dz \qquad 225 - \frac{225}{2}$$

$$= \underbrace{\cancel{15}z}_{75z-75/6 z^2} - \frac{1}{2}\left(5-\frac{5}{3}z\right)^3\left(-\frac{3}{5}\right) - \frac{25}{2}z^2 + \frac{25}{9}z^3 \Big|_0^3 = \left[\cancel{45} - 0 - \frac{25}{2}9 + 75 - \frac{3}{10}5^3\right]$$

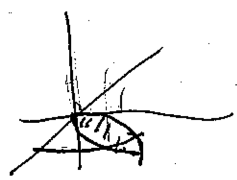
$$= 45 - 45\left(\frac{5}{2}\right) + \frac{75}{2}$$

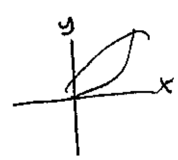
$$\frac{225}{2}$$

$$\frac{300}{2} - \frac{225}{2} = \boxed{\frac{75}{2}}$$

3. Find $\iiint_E (xy)\, dV$ where E is bound by $x=y^2$, $y=x^2$, $z=0$ and $z=x+y$.

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\, dz\, dy\, dx$$





$$= \int_0^1 \int_{x^2}^{\sqrt{x}} x^2y + xy^2\, dy\, dx$$

$$= \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3} \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^6}{2} - \frac{x^7}{3}\, dx$$

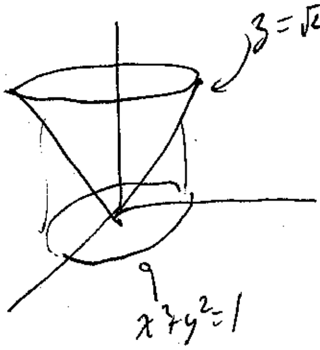
$$= \left[\frac{x^4}{8} + \frac{2}{7}\frac{x^{7/2}}{3} - \frac{x^7}{14} - \frac{x^8}{24}\right] = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{1}{12} - \frac{1}{42} = \frac{7}{84} - \frac{2}{84} = \boxed{\frac{5}{84}}$$

4. Convert

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{4x^2+4y^2}}(x+1)\,dz\,dy\,dx$$

to polar coordinates and solve. Also, give a (reasonable) sketch and/or description of the region of integration.



$$\int_0^{2\pi}\int_0^1\int_0^{2r}(r\cos\theta+1)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos\theta+r\ dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 2r^3\cos\theta+2r^2\,dr\,d\theta$$

$$=\int_0^{2\pi}\tfrac{1}{2}\cos\theta+\tfrac{2}{3}\,d\theta=4\pi/3$$

5. Find the volume in the first octant bound by the cylinder $x^2+y^2=4$ and the paraboloid $z=9-x^2-y^2$.

$$\iiint_E 1\,dV=\int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^2 9r-r^3\,dr\,d\theta$$

$$=\int_0^{\pi/2}\frac{9}{2}r^2-\frac{r^4}{4}\Big|_0^2$$

$$=\int_0^{\pi/2}18-4\,d\theta=14\ \pi/2=7\pi.$$

6. (Bonus) If you guess an answer to this question at random, the probability that you will be correct is:

(a) 25 % (b) 50 % (c) 66 % (d) 25 %