KEY

Math 225: Exam the First

You have two hours to take this closed-book, closed-note, and closed-colleague exam. You may use a basic calculator for arithmetic, trig functions, logarithms and exponentials, but no graphing or calculus functions.

1. Consider the parametric curve given by $x = -2t, y = 8t - 2t^2$

   (a) Find the slope of the tangent line when $t = 1$.

   $$\text{slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left.\frac{8-4t}{-2}\right|_{t=1} = \underline{-2}$$

   (b) Find the area bound by this curve and the x-axis. Be sure to explain any 'adjustments' you might need to make in your calculation.

   x axis: $y=0 \Rightarrow 8t-2t^2=0 \Rightarrow 2t(4-t)=0 \Rightarrow t=0,4$

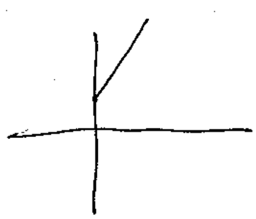
   $\frac{dx}{dt} \le 0 \Rightarrow$ curve runs right to left

   $$\Rightarrow \text{Area} = \int_4^0 (8t-2t^2)(-2)\,dt$$

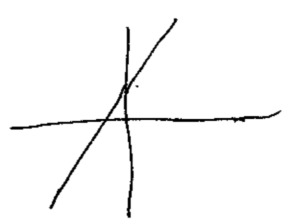
   $$= 2\left[4t^2 - \tfrac{2}{3}t^3\right]_0^4 = 2\left(64 - \tfrac{128}{3}\right) = \underline{\tfrac{128}{3}}$$

2. Explain the similarities and differences between the curves $\mathbf{r}(t) = \langle t^2, 2t^2+1\rangle$ and $\mathbf{r}(t) = \langle \frac{t-1}{2}, t\rangle$.

   Both satisfy $y = 2x+1$, so both are lines. However the first takes on only positive x-values, while the other takes on all x-values.



First



Second

3. Consider the points $A = (3, 1, 2)$, $B = (7, -1, 4)$ and $P = (x, y, z)$.

(a) Find the distance from A to B.

$$d(A,B) = \sqrt{(7-3)^2 + (-1-1)^2 + (4-2)^2}$$

$$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}.$$

(b) Write the following as a mathematical equation involving a dot product.

$\overrightarrow{AP} \perp \overrightarrow{BP}$

$\overrightarrow{AP} = \langle x-3, y-1, z-2 \rangle$

$\overrightarrow{BP} = \langle \cancel{y}x-7, y+1, z-4 \rangle$

$\overrightarrow{AP} \perp \overrightarrow{BP} \Rightarrow (x-3)(x-7) + (y-1)(y+1) + (z-2)(z-4) = 0$

(c) Simplify your equation into the form of a sphere and identify its center and radius.

↳ simplify

$x^2 - 10x + 21 + y^2 - 1 + z^2 - 6z + 8 = 0$

$x^2 - 10x + 25 + y^2 + z^2 - 6z + 9 = 4 + 1 + 1$

$(x-5)^2 + y^2 + (z-3)^2 = 6$

Center $= (5, 0, 3)$

radius $= \sqrt{6}$

(d) What relationships are there between the center, radius, and original points A and B?

Center = midpoint AB

radius = $\frac{1}{2}$ distance A to B → AB is a diameter of the sphere.

4. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Prove that if $\mathbf{a} \perp \mathbf{b}$ then $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

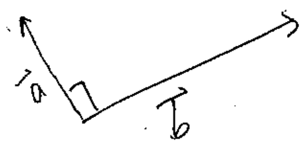
$$= \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \qquad \text{but } \vec{a}\cdot\vec{b} = \vec{0}$$

$$\Rightarrow \quad = |\vec{a}|^2 + 0 + |\vec{b}|^2$$

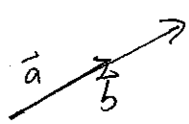
This is the pythagarean theorem

5. Give a (well-labeled) drawing of vectors $\mathbf{a}$ and $\mathbf{b}$ that satisfy each of the following equations.

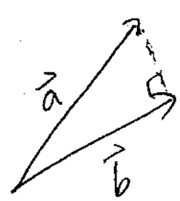
(a) $\mathbf{proj_b a} = \mathbf{0}$.



(b) $\mathbf{proj_b a} = \mathbf{a}$



(c) $\mathbf{proj_b a} = \mathbf{b}$.



6. Consider the points $A = (2, -1, 5)$, $B = (4, -3, 1)$, and $C = (1, -2, 3)$.

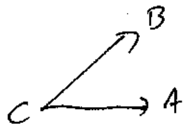
(a) Find the equation of the line through A and B.

point: $(2, -1, 5)$

dir vector: $\langle 2, -2, -4 \rangle$

line: $\vec{r}(t) = \langle 2, -1, 5 \rangle + t \langle 2, -2, -4 \rangle$.

(b) Find $\angle BCA$. You may leave your answer as an arccosine.



$\vec{CB} = \langle 3, -1, -2 \rangle$

$\vec{CA} = \langle 1, 1, 2 \rangle$

$$\theta = \arccos\left(\frac{\vec{CB} \cdot \vec{CA}}{|\vec{CB}||\vec{CA}|}\right) = \arccos\left(\frac{-2}{\sqrt{14}\sqrt{6}}\right)$$

(c) Find the equation of the plane containing $\triangle ABC$.

Plane: Point $(1, -2, 3)$

normal: $\vec{n} = \vec{CB} \times \vec{CA} = \langle 3, -1, -2 \rangle \times \langle 1, 1, 2 \rangle = \langle 0, -8, 4 \rangle$

plane: $0(x-1) - 8(y+2) + 4(z-3) = 0$

$-8y + 4z = 28$

$-2y + z = 7$

(d) Find the area of $\triangle ABC$.

Area $= \frac{1}{2} |\vec{CB} \times \vec{CA}| = \frac{\sqrt{80}}{2}$.

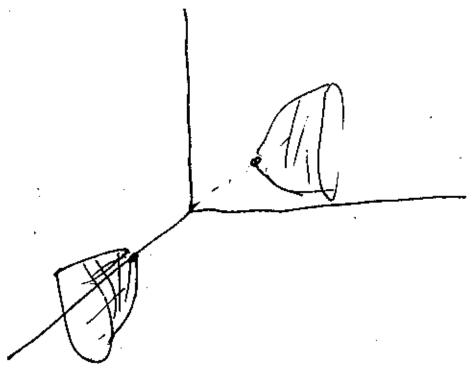
7. (a) Identify the following two surfaces. Draw a rough sketch of each (I will be generous on grading these. Make sure things point in the right directions):

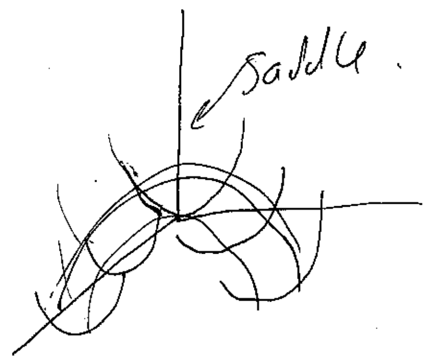
$$y^2 + z^2 + 1 = x^2$$

$$2z = x^2 - y^2$$

$y^2+z^2+1=x^2 \rightarrow$ hyperboloid of 2 sheets

$2z=x^2-y^2 \rightarrow$ hyperbolic paraboloid





(b) Find the intersection of the two surfaces. Show that they intersect in a plane. (You need not parametrize the intersection).

Intersection:

$x^2-y^2=2z$

$x^2-y^2=z^2+1$

$\rightarrow z^2+1=2z$

$z^2-2z+1=0$

$(z-1)^2=0$

$z=1$

$x^2-y^2=2$

Intersection is a hyperbola in the plane $z=1$

8. (a) Find the tangent line to $\mathbf{r}(t) = \langle t^2+1, t^3+t, 3t+1\rangle$ at the point when $t = 2$.

T. line: Point: $t=2$ $\vec{r}(2) = \langle 5, 10, 7\rangle$

dir $\vec{v}$: $\vec{r}\,'(2) = \langle 2t, 3t^2+1, 3\rangle\big|_{t=2}$

$= \langle 6, 13, 3\rangle$

line: $x = 5+6t$, $y = 10+13t$, $z = 7+3t$.

(b) Find the speed of $\mathbf{r}(t)$ when $t = 0$.

speed $= |\vec{r}\,'(0)| = |\langle 0, 1, 3\rangle| = \sqrt{10}$.

9. (a) Determine if the following lines are parallel, intersect, or are skew.

$$\mathbf{r}_1(t) = \langle 1, -1, 0\rangle + t\langle 1, 4, 2\rangle$$
$$\mathbf{r}_2(t) = \langle 2-t, 3+t, 1+3t\rangle$$

dir vectors:

$\langle 1, 4, 2\rangle$

$\langle -1, 1, 3\rangle$

Not parallel.

$x = 1+t = 2-s$

$y = -1+4t = 3+s$

$z = 2t = 1+3s$

solve x, y

$t = 1-s$

$-1+4(1-s) = 3+s$

$3+4s = 3+s \rightarrow s = 0$

$t = 1$

but for z

$2(1) \neq 1+3(0)$

So the lines are skew!

(b) If they are parallel or intersect, find the plane containing both of them. If they are skew, find a plane parallel to both of them through the origin.

Point $(0,0,0)$

$\vec{n} = \langle 1, 4, 2\rangle \times \langle -1, 1, 3\rangle$

$\langle 10, -5, 5\rangle$

plane: $10x - 5y + 5z = 0$.