KEY

Math 225: Exam the Second

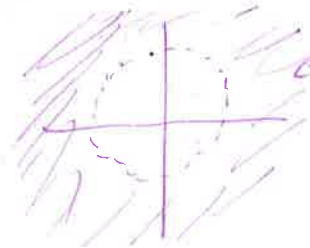
You have two hours to take this closed-book, closed-note, and closed-colleague exam. You may use a basic calculator for arithmetic, trig functions, logarithms and exponentials, but no graphing or calculus functions.

1. Let $f(x,y) = \frac{1}{\sqrt{x^2+y^2-1}}$

   (a) What is the domain of f? Give both an algebraic description and a sketch.

   Denominator cannot $= 0$.
   Radical term cannot be negative

   $\Rightarrow x^2+y^2-1 > 0 \quad , \quad x^2+y^2 > 1$

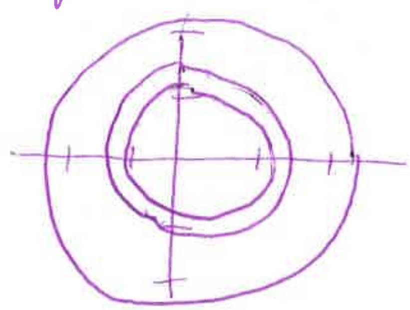
   

   (b) Draw level curves for f for values of $\frac{1}{2}, 1, 2$

   $K = \frac{1}{\sqrt{x^2+y^2-1}} \Rightarrow x^2+y^2 = 1 + \frac{1}{K^2}$

   level curves are circles of radius $\sqrt{1+\frac{1}{K^2}}$

f	$\frac{1}{2}$	1	2	$\to \infty$
radius	$\sqrt{5}$	$\sqrt{2}$	$\sqrt{\frac{5}{4}}$	$\to 1$

   (c) Are there level curves for $f \leq 0$? Explain.

   No. $\sqrt{x^2+y^2-1} > 0 \Rightarrow$ all values in the range are positive

2. Let $f(x, y) = x^2 + y^3 + 4y + x\cos(y)$

(a) Find the tangent plane at the point $(3, 0)$ and use it to approximate $f(\pi, 0.1)$

$f(3,0) = 9 + 0 + 0 + 3\cos 0 = 12$

$f_x = 2x + \cos y \xrightarrow{(3,0)} 6 + 1 = 7$

$f_y = 3y^2 + 4 + x\sin y \xrightarrow{(3,0)} 0 + 4 + 0 = 4$

T plane: $z - 12 = 7(x-3) + 4(y-0)$

$f(\pi, 0.1) \approx$

$12 + 7(3.14 - 3) + 4(0.1 - 0)$

$= 12 + 7(.14) + 4(0.1)$

$= \underline{13.38}$

(b) Find $D_{\mathbf{u}}f$ at $(3, 0)$ as we move towards $(2, 1)$.

$(3,0) \to (2,1) \xrightarrow{\vec{v}} \langle -1, 1 \rangle \xrightarrow{\text{unit}} \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\vec{u}}f = \vec{\nabla} f \cdot \vec{u} = \langle 7, 4 \rangle \cdot \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{-3}{\sqrt{2}}$

3. (a) Find the equation to the tangent plane to the surface $x^3y + y^3z + z^3x = 5$ at the point (2,1,-1).

$\vec{\nabla} F = \langle 3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3 \rangle \Big|_{(2,1,-1)} = \langle 11, 5, 7 \rangle$

T plane: $11(x-2) + 5(y-1) - 7(z+1) = 0$

(b) Find $\dfrac{\partial z}{\partial y}$ at this point.

$\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z} = \underline{\dfrac{-5}{7}}$

4. Let $f(x, y) = x^2 - 2x + 2y^2 - 4y + 5$.

(a) Find and classify the only critical point of f.

$f_x = 2\cancel{x} - 2 = 0$ $\quad$ c.p. $= (1,1)$

$f_y = 4y - 4 = 0$

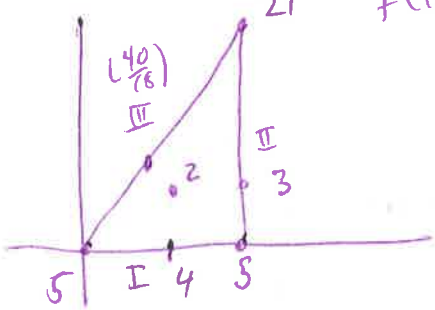
$D = (2)(4) - (0)^2 = 8 > 0$ so $(1,1)$ is a max or min

Since $f_{xx} > 0 \to$ graph is concave up $\to$ $(1,1)$ is a $\underline{\text{min}}$

(b) Find the maximum and minimum value of f on the triangle with vertices $(0,0), (2,4)$ and $(2,0)$

$(1,1)$ a candidate

$f(1,1) = 1-2+2-4+5 = 2$



I: $y = 0$

opt $f = x^2 - 2x + 5$

$f' = 2x - 2 \to x = 1$

$f = 1 - 2 + 5 = 4$

$f(0) = 5$

$f(2) = 5$

II $\quad x = 2$ $\quad f = 2y^2 - 4y + 5$

$f' = 4y - 4 = 0 \quad y = 1$

$f(1) = 2 - 4 + 5 = 3$

$f(0) = 5$

$f(4) = 32 - 16 + 5 = 21$

III $\quad y = 2x$

$f = x^2 - 2x + 8x^2 - 8\cancel{x} + 5 = 9x^2 - 10x + 5$

$f' = 18x - 10 = 0 \quad x = \frac{10}{18}$

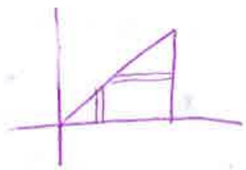
$f\left(\frac{10}{18}\right) = \frac{900}{18^2} - \frac{100}{18} + 5$

$\frac{900}{18^2} - \frac{1800}{18^2} + \frac{1620}{18^2} = \frac{720}{18^2} = \frac{40}{18}$

5. Calculate

$$\int_0^3 \int_y^3 e^{x^2}\, dx\, dy$$

by first reversing the order of integration.

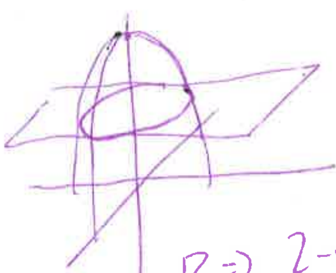


$$\int_0^3 \int_0^x e^{x^2} dy\, dx = \int_0^3 x e^{x^2} dx$$

$$= \left[\frac{e^{x^2}}{2} \right]_0^3$$

$$= \left[\frac{e^9 - 1}{2} \right]$$

6. Find the volume bound by the paraboloid $z = 2 - x^2 - y^2$ and $z = 1$.



$$\iint_R (2 - x^2 - y^2) - 1\, dA$$

$R \Rightarrow 2 - x^2 - y^2 = 1$

$x^2 + y^2 = 1$

unit circle

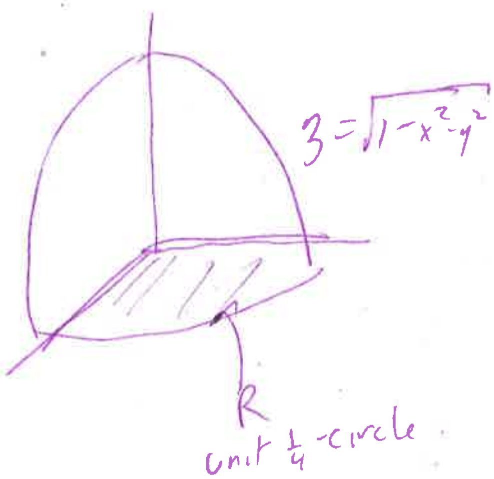
$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4} \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \boxed{\pi}$$

7. Find the volume of the unit sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant using *calculus* (You may check your answer if you remember the formula for the volume of a sphere!).



$$\iint_R \sqrt{1-x^2-y^2}\, dA$$

$$= \int_0^{\pi/2}\int_0^1 \left(\sqrt{1-r^2}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{3}(1-r^2)^{3/2}\right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{2}{3}\, d\theta = \frac{\pi}{6}$$

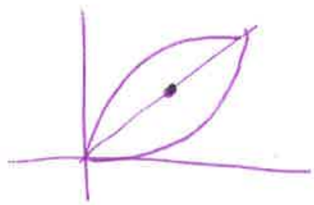
8. Set up, but don't compute, the integral for the surface area of $f(x, y) = \sqrt{x^2 + y^2 - 1}$ over $[3, 4] \times [4, 6]$.

$$f_x = \frac{1}{2}(x^2+y^2-1)^{-1/2}(2x) \qquad (f_x)^2$$

$$f_y = \frac{1}{2}(x^2+y^2-1)^{1/2}(2y) \qquad (f_y)^2$$

$$S.A = \int_3^4 \int_4^6 \sqrt{1 + \frac{x^2}{x^2+y^2-1} + \frac{y^2}{x^2+y^2-1}}\, dy\, dx.$$

9. Find the center of mass of a plate in the shape of the area between the curves $y = x^2$ and $x = y^2$ in the first octant, if the density of the plate is given by the function $\rho(x, y) = xy$. (Hints: You may save yourself some work by employing symmetry. The mass of this plate is $\frac{1}{12}$).



NB: $\bar{x} = \bar{y}$

$$\bar{x} = \frac{1}{mass} \iint_R x\rho(x,y)\, dA = 12 \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy\, dx$$

$$= 12 \int_0^1 \frac{x y^2}{2} \Big|_{x^2}^{\sqrt{x}} dx$$

$$= 12 \int_0^1 \frac{x^3}{2} - \frac{x^6}{2} \, dx = 6 \left[\frac{x^4}{4} - \frac{x^7}{7} \right]_0^1$$

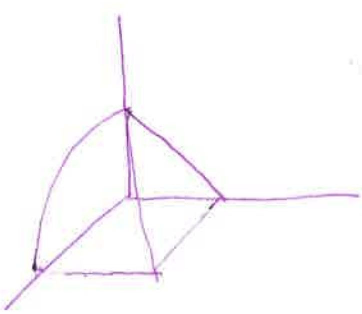
$$= \frac{6 \cdot 3}{28} = \frac{9}{14}$$

C.O.M $= \left(\frac{9}{14}, \frac{9}{14}\right)$

10. Set up the bounds for

$$\iiint_E f(x, y, z)\, dV,$$

where E is bound by $x = 0, y = 0, z = 0, x = 1 - z^2$, and $y + z = 1$.



$$\int_0^1 \int_0^{1-z} \int_0^{1-z^2} f(x, y, z)\, dx\, dy\, dz$$