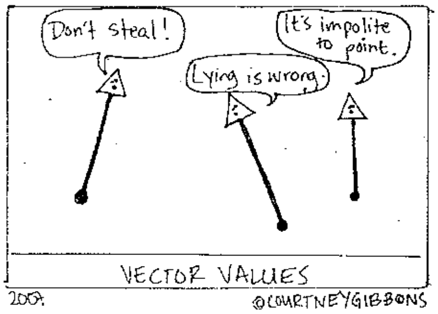
KEY

Math 225: Quiz the Second
February 7, 2014

This quiz is closed book and closed notes. Please justify all of your answers. You have until the end of the class period to finish.



1. Let **a**, **b**, and **c** be vectors. For each quantity, state whether it is a vector, a scalar, or nonsense.

   (a) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ → vector
   vec. × vec

   (b) $(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}$ → nonsense
   sc. + vec

   (c) $\frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$ → scalar
   sc(vec) · vec

   (d) $\mathbf{a}^2$ → nonsense (no operation is given here)

2. Consider the points $A = (0, 1, 2)$, $B = (2, 2, 1)$ and $C = (-3, 4, -1)$

(a) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{CB}$.

$\overrightarrow{AB} = \langle 2, 1, -1 \rangle$ $\qquad$ $\overrightarrow{CB} = \langle 5, -2, 2 \rangle$

$\overrightarrow{AC} = \langle -3, 3, -3 \rangle$

(b) Is angle $\angle BAC$ acute, right, or obtuse? Explain.

$\measuredangle BAC = \theta$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \frac{(-6+3+3)}{\sqrt{6}\cdot 3\sqrt{3}} = 0$$ so $\measuredangle BAC$ is $\underline{\text{RIGHT}}$

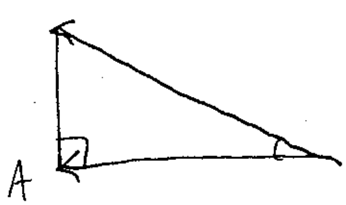
(c) Find a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\begin{array}{rl} \overrightarrow{AB} = & \langle 2, 1, -1 \rangle \\ \times\ \overrightarrow{AC} = & \langle -3, 3, -3 \rangle \\ \hline & \langle -3-(-3), 3-(-6), 6-(-3) \rangle = \underline{\langle 0, 9, 9 \rangle} \end{array}$$

(d) Project $\overrightarrow{CB}$ onto $\overrightarrow{CA}$. What do you notice about your answer and why?

$$\text{Proj}_{\overrightarrow{CA}}\overrightarrow{CB} = \frac{(\overrightarrow{CA}\cdot\overrightarrow{CB})}{|\overrightarrow{CA}|^2}(\overrightarrow{CA}) = \frac{\langle 3,-3,3\rangle\cdot\langle 5,-2,2\rangle}{(3^2+(-3)^2+(3)^2)}\overrightarrow{CA} = \frac{(15+6+6)}{27}\overrightarrow{CA}$$

$$= \overrightarrow{CA} = \langle -3, 3, -3 \rangle$$



Since $\measuredangle BAC$ is a right angle, the projection equals the initial vector.

3. Let $\mathbf{v} = \langle 2, 1, -2\rangle$, and let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ be a unit vector.

(a) Give two formulas that relate $\mathbf{u}$, $\mathbf{v}$, and the angle between them.

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta$$

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

(b) What is the maximum value of $\mathbf{u} \cdot \mathbf{v}$? Which vector $\mathbf{u}$ gives that maximum?

$$|\vec{v}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

$$= 1 \cdot 3 \cos\theta$$

max value $= 3$, when $\theta = 0$, $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \rangle$

(c) What is the minimum value of $\mathbf{u} \cdot \mathbf{v}$? Which vector $\mathbf{u}$ gives that minimum?

min value $= -3$ when $\theta = \pi$, $\vec{u} = \frac{-\vec{v}}{|\vec{v}|} = \langle \frac{-2}{3}, \frac{-1}{3}, \frac{2}{3} \rangle$

4. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are the same length, then $(\mathbf{a} + \mathbf{b})$ is orthogonal to $(\mathbf{a} - \mathbf{b})$

If $|\vec{a}| = |\vec{b}|$

Then $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$

$$= \vec{a} \cdot \vec{a} + \cancel{\vec{b} \cdot \vec{a}} - \cancel{\vec{a} \cdot \vec{b}} - \vec{b} \cdot \vec{b}$$

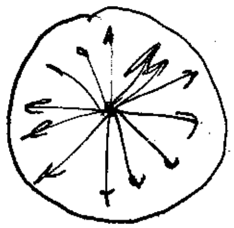
$$= |\vec{a}|^2 - |\vec{b}|^2 = 0$$

Hence $\vec{a} + \vec{b} \perp \vec{a} - \vec{b}$.

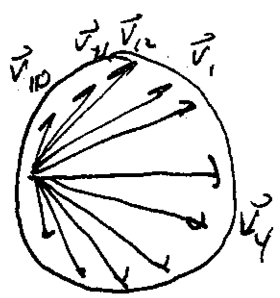
5. (Bonus)

(a) The face of a clock has 12 vectors, each with their tail at the center, and each with their head on a different number. What is the sum of these vectors.

(b) The face of the clock has 11 vectors, each with their tail on '9', and each pointing to a different number. What is the sum of these vectors?



$\sum_{\text{vectors}} = \vec{0}$ since each vector has an opposite.



Let $\vec{v}_7$ point from 9 to 7 o'clock.

Here each pair of vectors

$$\vec{v}_{10} + \vec{v}_4 = \vec{v}_3$$

$$\vec{v}_{11} + \vec{v}_5 = \vec{v}_3$$

etc

so $\sum_{\text{vectors}} = 6 \cdot \vec{v}_3$.