KEY

Math 225: Exam the First

February 20, 2015

You have two hours to take this closed-book, closed-note, and closed-colleague exam. You may use a basic calculator for arithmetic, trig functions, logarithms and exponentials, but no graphing or calculus functions.

1. Draw each of the following parametric curves, each with attention to direction and allowable x and y values. All graphs are defined for all real t.

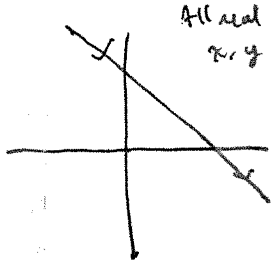
   (a) $\mathbf{r}(t) = \langle t, 1-t\rangle$

   (b) $\mathbf{r}(t) = \langle -e^t, 1+e^t\rangle$

   (c) $\mathbf{r}(t) = \langle \sin^2(t), \cos^2(t)\rangle$

   $\Big\}$ All three satisfy $x + y = 1$

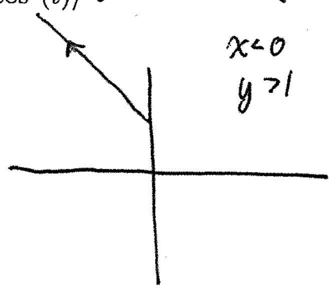
All real x, y



a

$x<0$

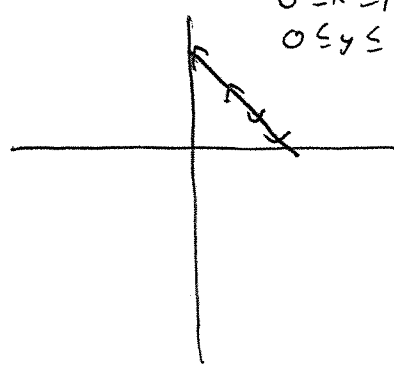
$y>1$



b

$0 \le x \le 1$

$0 \le y \le 1$



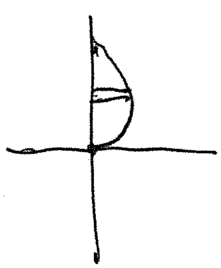
c

2. Consider the curve $x = 4t - t^2$, $y = 3t$.

   (a) Find the points (if they exist) where the curve has a vertical or horizontal tangent.

$\dfrac{dy}{dx} = \dfrac{3}{4-2t}$ H-tangent never

V-tangent: $4-2t=0 \Rightarrow t=2$ points $(4(2)-4, 3(2)) = (4,6)$



   (b) Find the area bound by this curve and the y-axis.

$x=0$ @ $t=0, 4$

$$\text{Area} = \int x\,dy = \int_0^4 4t - t^2 (3)\,dt = 3\left[2t^2 - \frac{t^3}{3}\right]_0^4$$

$$= 3\left[32 - \frac{64}{3}\right] = 3\left(\frac{32}{3}\right) = 32$$

   (c) Set up, but don't compute, the integral for the length of the curve that sits to the right of the y-axis.

$$\text{Length} = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^4 \sqrt{(4-2t)^2 + 9}\,dt.$$

3. Consider the points $A = (1, 4, 6)$, $B = (3, -2, 8)$ and $P = (x, y, z)$.

(a) Write the following as a mathematical equation.

$d(A, P) = d(B, P)$

$$\sqrt{(x-1)^2+(y-4)^2+(z-6)^2} = \sqrt{(x-3)^2+(y+2)^2+(z-8)^2}$$

(b) Simplify your equation into the form of a plane and give (and comment on!) its normal vector.

$$x^2-2x+1+y^2-8y+16+z^2-12z+36 = x^2-6x+9+y^2+4y+4+z^2-16z+64$$

(All square terms) Cancel!

$$4x - 12y + 4z = 24$$

$$\vec{n} = \langle 4, -12, 4 \rangle = 2\overrightarrow{AB} = 2\langle 2, -6, 2 \rangle$$

(c) Give the intersection point of this plane with each of the coordinate axes.

x-axis $\quad y = z = 0 \quad \to \quad (6, 0, 0)$

y-axis $\quad x = z = 0 \quad \to \quad (0, -2, 0)$

z-axis $\quad y = x = 0 \quad \to \quad (0, 0, 6)$

4. Consider the points $A = (3, 1, 0)$, $B = (4, -1, 2)$, and $C = (5, 3, 1)$.

(a) Find the equation of line through A that is parallel to the line through B and C.

Point $(3,1,0)$

direction $\to$ $\vec{BC} = \langle 1, 4, -1 \rangle$

line: $\langle 3, 1, 0 \rangle + t\langle 1, 4, -1 \rangle$

(b) Show that $\triangle ABC$ is a right triangle.

$\vec{AB} = \langle 1, -2, 2 \rangle$

$\vec{BC} = \langle 1, 4, -1 \rangle$

$\vec{AC} = \langle 2, 2, 1 \rangle$

$\vec{AB} \cdot \vec{AC} = \langle 1, -2, 2 \rangle \cdot \langle 2, 2, 1 \rangle$

$= 2 - 4 + 2 = 0$

So $\measuredangle A = 90°$, ABC is a right triangle

(c) Find the area of $\triangle ABC$ by

i. Cross-Products

$$\begin{array}{rl} \vec{AB} & \langle 1, -2, 2 \rangle \\ \times\ \vec{AC} & \langle 2, 2, 1 \rangle \\ \hline & \langle -2-4,\ 4-1,\ 2- -2 \rangle \\ & \langle -6, 3, 6 \rangle \end{array}$$

Area of $\triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}\sqrt{36 + 9 + 36} = \frac{9}{2}$

ii. High School Geometry

$|\vec{AB}| = 3$

$|\vec{AC}| = 3$

Area $= \frac{1}{2}(3 \cdot 3) = \frac{9}{2}$

5. Prove that if **a** and **b** are vectors such that **a** + **b** and **a** − **b** are orthogonal, then |**a**| = |**b**|.

$(\vec{a}+\vec{b}) \perp (\vec{a}-\vec{b})$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$

$\Rightarrow (\vec{a}\cdot\vec{a}) + (\vec{b}\cdot\vec{a}) - (\vec{a}\cdot\vec{b}) - (\vec{b}\cdot\vec{b}) = 0$

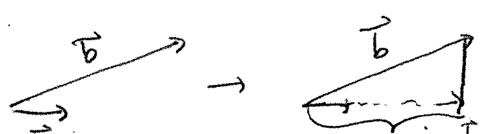
$\Rightarrow |\vec{a}|^2 + 0 - |\vec{b}|^2 = 0$

$|\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{a}| = |\vec{b}|$

6. (a) Can it be the case that $\mathbf{proj_a b}$ is longer than **a**? Give (draw) an example or explain why not.

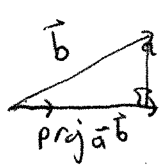
Yes...



$|\text{proj}_{\vec{a}}\vec{b}| > |\vec{a}|$

(b) Can it be the case that $\mathbf{proj_a b}$ is longer than **b**? Give (draw) an example or explain why not.

No.



Note that $\vec{b}$ is the hypotenuse of a right triangle with $\text{proj}_{\vec{a}}\vec{b}$ as one of the legs.

7. (a) Find the equation of the plane through the point $(1, -1, -1)$ that is parallel to the plane $5x - y - z = 6$.

$\vec{n} = \langle 5, -1, -1 \rangle$

point: $(1, -1, -1)$

plane

$5(x-1) + -1(y+1) - 1(z+1) = 0$

$5x - y - z = 7.$

(b) Find the line of intersection of the plane you found in part (a) with the plane $x+y-z = 1$.

line vector

$\vec{n}_1 = \langle 5, -1, -1 \rangle$

$\vec{n}_2 = \langle 1, +1, -1 \rangle$

$\langle 1 - -1, -1+5, 5+1 \rangle$

$= \langle 2, 4, 6 \rangle$

point

$5x - y - z = 7$

$x + y - z = 1$

Choose $x = 0$

$-y - z = 7$

$y - z = 1$

$-2z = 8$

$z = -4$

$y = -3$

point $= (0, -3, -4)$

line $= \langle 0, -3, -4 \rangle + t \langle 2, 4, 6 \rangle$

8. Match the equation to the surface description. Warning: There is one outlier in each group!!

(a) $x^2 + 4y^2 + 16 = 16z$ IV

(b) $x^2 + 4y^2 + 16z^2 = 16$ III

(c) $x^2 - 4y^2 + 16 = 16$ none

(d) $x^2 + 4y^2 + 16 = 16z^2$ II

I Hyperboloid of One Sheet

II Hyperboloid of Two Sheets

III Ellipsoid

IV Elliptical Paraboloid

9. Let $\mathbf{r}(t) = \langle \sin(2t), e^t + 2, t^3 + t + 3 \rangle$

(a) Find a unit vector tangent to $\mathbf{r}(t)$ when $t = 0$.

$\vec{r}'(t) = \langle 2\cos(2t), e^t, 3t^2+1 \rangle$

$\vec{r}'(0) = \langle 2, 1, 1 \rangle$ unit vector $= \langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$

(b) Find the tangent line to $\mathbf{r}(t)$ when $t = 0$.

line: $\vec{r}'(0) = \langle 2, 1, 1 \rangle$

$\vec{r}(0) = \langle 0, 3, 3 \rangle$

line: $x = 2t$
$y = 3 + t$
$z = 3 + t$

(c) If $\mathbf{r}(t)$ is the curve of an object in motion, is that object ever at 'rest'? Explain.

No. Note that $\vec{r}'(t) = \langle 2\cos 2t, e^t, 3t^2+1 \rangle$
so it is always moving in either the y or z direction.

10. (Bonus) Tell me about a panel (or other activity) that you attended in conjunction with yesterday's Power and Privilege Symposium.