KEY

Math 225: Exam the Second April 17, 2015

You have two hours to complete this closed-book, closed-note, closed colleague exam. You may use a calculator for arithmetic only (trig functions and exponentials are okay, but no plotting and no calculus).

1. Let $f(x,y) = \frac{1}{\sqrt{x^2+y^2+1}}$

   (a) What is the domain of f?

   All real x, y $(x^2+y^2+1 > 0$ always$)$

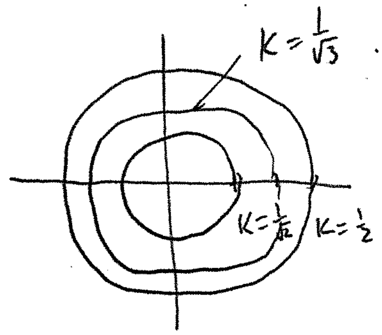
   (b) What is the *range* of f? That is, for which values k can we draw level curves for f?

   $K = \frac{1}{\sqrt{x^2+y^2+1}} \Rightarrow x^2+y^2 = \frac{1}{K^2} - 1 \geq 0$

   $\Rightarrow K \leq 1,\ K > 0.$

   (c) Draw and label at least three level curves for f, with attention to spacing.

   Circles, centered @ the origin



2. (a) Find the equation of the tangent plane to $f(x, y) = \ln(x - 2y)$ at the point $(3, 1)$.

$f_x = \frac{1}{x-2y}$ @ $(3,1) = 1$

$f_y = \frac{-2}{x-2y}$ @ $(3,1) = -2$

$f(3,1) = \ln(3-2) = 0$

Tplane: $z = (x-3) - 2(y-1)$

(b) At the point $(3, 1)$, in what direction should you move to increase the fastest?

$\vec{\nabla} f = \langle 1, -2 \rangle$ move in that direction

Direction $= \langle 1, -2 \rangle$.

3. For a (well-behaved) function of two variables, there are four 'second derivatives'. Clairaut's theorem says two of them are the same, thus there are really only three distinct 'second derivatives'.

(a) How many 'third derivatives' are there, and how many are distinct?

There are

8 3rd Devs, 4 are Distinct

(b) How many 'fourth derivatives' are there, and how many are distinct?

There are

16 4th Devs, 5 are Distinct

(c) How many 'nth derivatives' are there, and how many are distinct?

There are

2^n nth devs

(each step 2 choices $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$)

$n+1$ are distinct

(we only care about how many x's, y's there are).

6. Find

$$\iint_R (x+1)\cos(xy+y)$$

on $[0,2]\times[0,1]$

$$\int_0^2\int_0^1 (x+1)\cos(xy+y)\,dy\,dx$$

$u = xy+y$

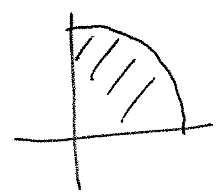
$du = x+1\ dy$

$$\int_0^2 \sin(xy+y)\Big|_0^1 dx = \int_0^2 (\sin(x+1))\,dx = -\cos(x+1)\Big|_0^2$$

$$= -\cos 3 + \cos 1$$

7. Find

$$\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$$



Polars!

$$\int_0^2\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$$

$$= \frac{\pi}{2}\int_0^2 e^{-r^2} r\,dr = \frac{\pi}{2}\left(-\frac{1}{2}e^{-r^2}\Big|_0^2\right)$$

$$= -\frac{\pi}{4}\left(e^{-4}-1\right)$$

4. Find and classify the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$. (There are 2)

$f_x = 3x^2 - 3y = 0 \qquad y = x^2$

$f_y = 3y^2 - 3x = 0$

$3y^2 - 3x = 3x^4 - 3x = 0 \qquad$ cp's are $(0,0)$ $(1,1)$

$3x(x^3 - 1) = 0$

$x = 0, 1$

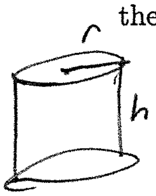
$y = 0, 1$

$D = (6x)(6y) - 3^2 = 36xy - 9$

$D_{(0,0)} = -9 \qquad (0,0)$ is a saddle point.

$D_{(1,1)} = 27 \qquad (1,1)$ is a max or $\boxed{\text{min} \quad f_{xx} > 0}$

5. A cylindrical can is to have surface area 54π square inches. Find the dimensions that maximize the ~~area~~ volume of the can. You may use any method that you wish.



$SA = 2\pi r^2 + 2\pi r h = 54\pi \rightarrow r^2 + rh = 27$

$\Rightarrow h = \frac{27 - r^2}{r}$

$\text{Volume} = \pi r^2 h$

$V = \pi r^2 \left(\frac{27 - r^2}{r}\right)$

$V = \pi (27r - r^3)$

$V'(r) = \pi(27 - 3r^2) = 0$

$r = 3 \Rightarrow h = \frac{27 - 9}{3} = \frac{18}{3} = 6$

Dimensions are $r = 3$ $h = 6$

height = Diameter $\rightarrow$ can is "Squarish"

10. (a) Give the polar coordinates equation of the circle $x^2 + y^2 = 4y$

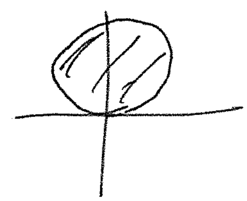
$x^2+y^2=4y$

$\Rightarrow r^2 = 4r\sin\theta$

$\Rightarrow r = 4\sin\theta$

(b) Find the surface area of the plane $3x + 5y + z = 15$ inside the ~~cone~~ cylinder $x^2 + y^2 = 4y$. Note: You should be able to do this one without any integration.

$\hookrightarrow D$



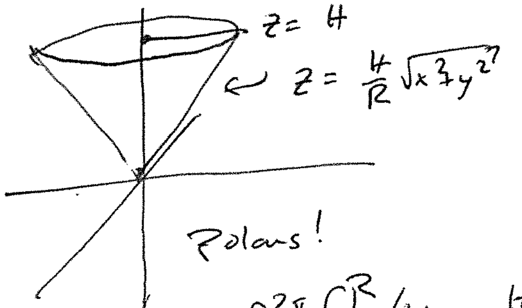
$$S.A = \iint_D \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

$$= \int_0^{\pi}\int_0^{4\sin\theta} \sqrt{1+9+25}\, dA$$

$$= \sqrt{35}\iint_D 1\, dA = \sqrt{35}\,(\text{Area}(D))$$

$$= \sqrt{35}\,(4\pi).$$

8. Find (using Calculus!) the volume of a cone of height H and radius R. Do this by finding the volume bound below by the cone $z = \frac{H}{R}\sqrt{x^2+y^2}$ and the plane $z = H$.



Polars!

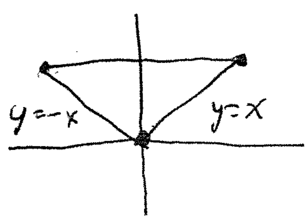
$$\int_0^{2\pi}\int_0^R \left(H - \frac{H}{R} r\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^R Hr - \frac{H}{R} r^2\,dr\,d\theta$$

top - bottom

$$= \int_0^{2\pi} \frac{HR^2}{2} - \frac{HR^3}{3R}\,d\theta$$

$$= \int_0^{2\pi} \frac{HR^2}{6}\,d\theta = \boxed{\frac{1}{3}\pi H R^2}$$

9. Find the center of mass of a plate in the shape of a triangle with vertices (0,0), (-1,1), (1,1) whose density is given by $\rho(x,y) = x^2 y$. You may employ symmetry where it is useful.



$$\text{mass} = \iint_{\text{plate}} x^2 y\,dA = \int_0^1\int_{-y}^{y} x^2 y\,dx\,dy$$

$$= \int_0^1 \frac{x^3}{3} y \Big]_{-y}^{y} dy$$

$$= \frac{1}{3}\int_0^1 \frac{y^4}{\cancel{3}} + \frac{y^4}{\cancel{3}} = \frac{2}{3}\int_0^1 y^4 = \frac{2}{3}\frac{y^5}{5}\Big]_0^1$$

$$= 2/15$$

$\bar{x} = 0$ by symmetry

$$\bar{y} = \frac{15}{2}\int_0^1\int_{-y}^{y} x^2 y^2\,dx\,dy = \frac{15}{2}\int_0^1 \frac{2}{3} y^5\,dy = \frac{15}{2}\cdot\frac{2}{3}\cdot\frac{1}{6} = \boxed{5/6}$$

C.o.m. $= (0, 5/6)$