Practice Exam Key February 18, 2015

1. $x = -2t \quad y = 8t - 2t^2$

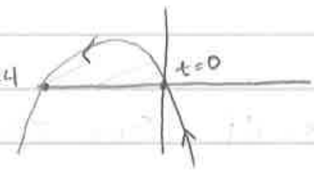
(a) Slope of tangent when $t=1$

$$\text{Slope} = \frac{dy}{dx} = \left(\frac{dy}{dt} \Big/ \frac{dx}{dt}\right) = \left.\frac{8-4t}{-2}\right|_{t=1} = \frac{4}{-2} = -2$$

(b) Area bound by curve and x-axis $(y=0)$

$8t - 2t^2 = 0$

$t = 0$ or $t = 4$



when $t=0$, $x=0$

when $t=4$, $x=8$ As $t \uparrow$, y gets larger and more negative

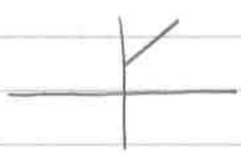
$$\text{Area} = \int y\,dx = \int_4^0 (8t - 2t^2)(-2)\,dt = 2\int_0^4 8t - 2t^2\,dt$$

$$= 2\left[4t^2 - \frac{2}{3}t^3\right]_0^4 = 2\left[64 - \frac{128}{3}\right] = \frac{128}{3}$$

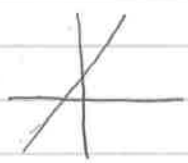
2. A: $\vec{r}(t) = \langle t^2, 2t^2+1\rangle$ B: $\vec{r}(t) = \left\langle \frac{t-1}{2}, t\right\rangle$ C: $\vec{r}(t) = \langle \sin t, 2\sin t + 1\rangle$

All are variants of $y = 2x+1$

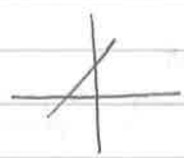
A



B



C



only positive outputs all outputs $-1 \le x \le 1 \quad -1 \le y \le 3$

3. $A(3,1,2) \quad B(7,-1,4) \quad P(x,y,z)$

(a) d A to B $= \sqrt{(7-3)^2 + (-1-1)^2 + (4-2)^2} = \sqrt{16+4+4} = \sqrt{24}$

(b) $\vec{AP} \perp \vec{BP} \quad \vec{AP} = \langle x-3, y-1, z-2\rangle \quad \vec{BP} = \langle x-7, y+1, z-4\rangle$

$\langle x-3, y-1, z-2\rangle \cdot \langle x-7, y+1, z-4\rangle = 0$

$(x-3)(x-7) + (y-1)(y+1) + (z-2)(z-4) = 0$

$x^2 - 10x + 21 + y^2 - 1 + z^2 - 6z + 8 = 0$

(c) $x^2 - 10x + 25 + y^2 + z^2 - 6z + 9 = -28$ Complete the square

$(x-5)^2 + y^2 + (z-3)^2 = 6$

Center at $(5,0,3)$ radius $= \sqrt{6}$

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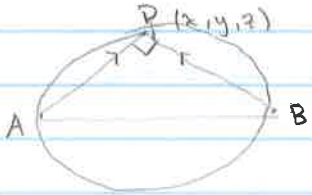
3. (d) The center is the midpoint between A and B

Radius is halfway between the two points $\quad \sqrt{6} = \frac{\sqrt{24}}{2}$



4. $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ Prove $\vec{a} \perp \vec{b}$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

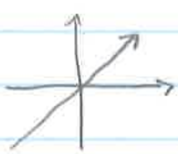
$$\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$4\,\vec{a}\cdot\vec{b} = 0$$

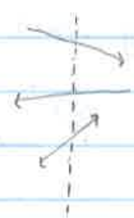
$$\vec{a}\cdot\vec{b} = 0 \quad \text{so} \quad \vec{a} \perp \vec{b}$$

Proves diagonals of a rectangle are equal

5. There might be a plane parallel to all 3. There may be a plane that is not parallel to all 3.

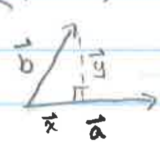


No parallel plane



parallel plane

6. $\vec{a} = \langle 1,3,4 \rangle \quad \vec{b} = \langle 1,0,-3 \rangle \quad \vec{x}, \vec{y}$ where $\vec{x} \parallel \vec{a}$, $\vec{y} \perp \vec{a}$, $\vec{x}+\vec{y} = \vec{b}$



$$\vec{x} = \text{proj}_{\vec{a}}\,\vec{b}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

$$= \frac{1+0-12}{26}\langle 1,0,-3 \rangle = \left\langle -\frac{11}{26}, \frac{33}{26}, \frac{44}{26} \right\rangle = \vec{x}$$

$$\vec{y} = \vec{b} - \vec{x}$$

$$\vec{y} = \left\langle \frac{37}{26}, \frac{33}{26}, -\frac{34}{26} \right\rangle$$

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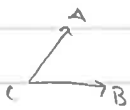
7. $A(2,-1,5)$ $B(4,-3,1)$ $C(1,-2,3)$

(a) Line through A and B point $= (2,-1,5)$

direction $= \langle 2,-2,-4\rangle$

$\vec{l}(t) = \langle 2,-1,5\rangle + t\langle 2,-2,-4\rangle$

(b) 

$\vec{CB} = \langle 3,-1,-2\rangle$ $\vec{CA} = \langle 1,1,2\rangle$

$$\theta = \arccos\left(\frac{\vec{CB}\cdot\vec{CA}}{|\vec{CB}||\vec{CA}|}\right)$$

$$\theta = \arccos\left(\frac{-2}{\sqrt{14}\sqrt{6}}\right)$$ obtuse angle

(c) Plane containing ΔABC (need point and normal)

$\vec{CA}\times\vec{CB}$

$$\begin{array}{r} \langle 1 \quad 1 \quad 2\rangle \\ \times \langle 3 \quad -1 \quad -2\rangle \\ \hline \langle 0,\ 8,\ -4\rangle \end{array} = \text{vector } \vec{n}$$

Plane: $0(x-1) + 8(y+2) - 4(z-3) = 0$

$8y - 4z = -28$

$2y - z = -7$

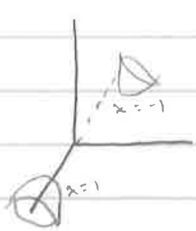
Either works

(d) Area $= \frac{1}{2}|\vec{CA}\times\vec{CB}| = \frac{\sqrt{80}}{2}$

8. $y^2 + z^2 + 1 = x^2$

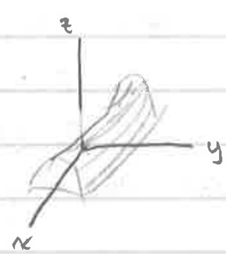
Hyperboloid of 2 sheets

$x \geq 1$ or $x \leq -1$



$2z = x^2 - y^2$

Hyperbolic paraboloid



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(b) Intersection $z^2 + 1 = x^2 - y^2$

$z^2 + 1 = 2z$

$z^2 - 2z + 1 = 0 \longrightarrow z = 1$

$z = x^2 - y^2$ hyperbola

9. (a) $\vec{r}(t) = \langle t^2+1, t^3+t, 3t+1 \rangle$ at $t = 2$

point $= (5, 10, 7)$

direction vector $\vec{r}\,'(2) = \langle 2t, 3t^2+1, 3 \rangle = \langle 4, 13, 3 \rangle$

$\vec{\ell}(t) = \langle 5+4t, 10+13t, 7+3t \rangle$

(b) speed at $t=0$ $\langle 2t, 3t^2+1, 3 \rangle\big|_{t=0} = \left| \langle 0, 1, 3 \rangle \right| = \sqrt{10}$

10. $x = 1+t = 2-s$ $\quad y = -1+t = 3+s$ $\quad z = 2t = 1+3s$

Solve for $t = 1$, $s = 0$

but then $z = 2 \neq 1+3 = 0$ The lines are skew

point $\langle 0,0,0 \rangle$ $\quad \vec{n} = \langle 1, 4, 2 \rangle$

$\langle -1, 1, 3 \rangle$

$\vec{n} = \langle 10, -5, 5 \rangle$

plane: $10x - 5y + 5z = 0$

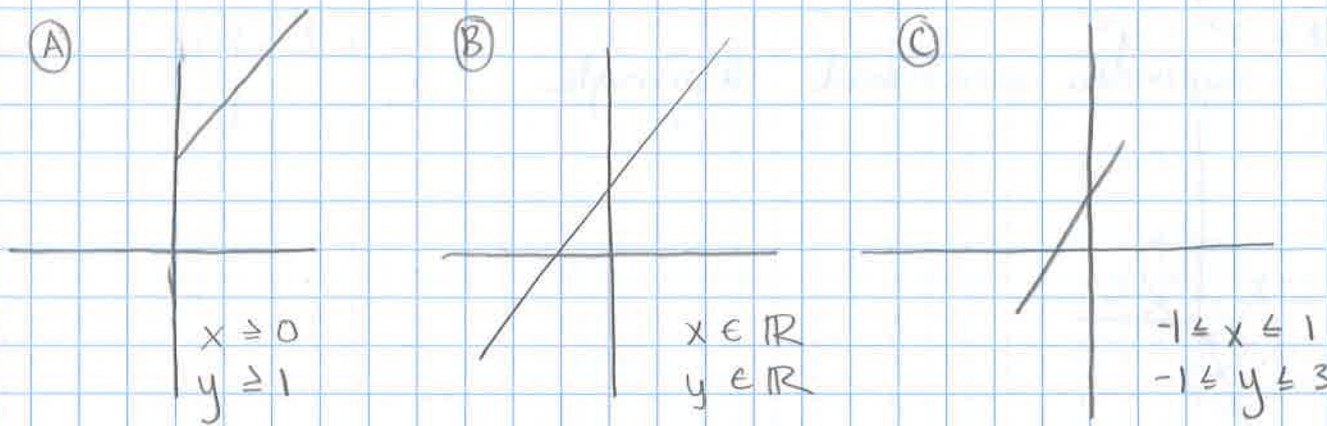
Midterm Practice Key: Even 2-18-1

2. $\text{curve}_A = \langle t^2, 2t^2+1 \rangle$

$\text{curve}_B = \langle \frac{t-1}{2}, t \rangle$

$\text{curve}_C = \langle \sin t, 2\sin t + 1 \rangle$

$\Big\}$ all variants of $y = 2x+1$



Different portions of same graph

4. $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ prove orthogonal (dot product = 0)

$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$(\vec{a}+\vec{b})\circ(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\circ(\vec{a}-\vec{b})$

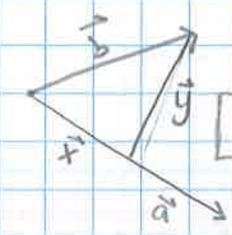
$\cancel{\vec{a}\circ\vec{a}} + \cancel{\vec{b}\circ\vec{b}} + \vec{a}\circ\vec{b} + \vec{a}\circ\vec{b} = \cancel{\vec{a}\circ\vec{a}} + \cancel{\vec{b}\circ\vec{b}} - \vec{a}\circ\vec{b} - \vec{a}\circ\vec{b}$

$4\,\vec{a}\circ\vec{b} = 0$

$\Rightarrow \vec{a}\circ\vec{b} = 0$

$\vec{a} \perp \vec{b}$ ✓

6.
$\vec{a} = \langle 1, 3, 4 \rangle$
$\vec{b} = \langle 1, 0, -3 \rangle$
$\vec{x}, \vec{y}$
$\vec{x} \parallel \vec{a}$
$\vec{y} \perp \vec{a}$
$\vec{x} + \vec{y} = \vec{b}$



projection!

$\Big[\vec{x} = \text{proj}_{\vec{a}}\vec{b} = \frac{(\vec{a}\circ\vec{b})\,\vec{a}}{|\vec{a}|^2} = \frac{1+0-12}{26}\langle 1, 3$

$\langle 1, 0, -3 \rangle = \vec{b}$

$- \langle \frac{-11}{26}, \frac{-33}{26}, \frac{-44}{26} \rangle = \vec{x}$

$\langle \frac{37}{26}, \frac{33}{26}, \frac{-34}{26} \rangle = \vec{y}$

8. $y^2 + z^2 + 1 = x^2$ 2-18-15

hyperboloid of two sheets (action on x ◡̈)

$2z = x^2 - y^2$

hyperbolic parabuloid #pringle

intersection

$$z^2 + 1 = x^2 - y^2$$
$$2z = x^2 - y^2$$
$$z^2 + 1 = 2z$$
$$z^2 - 2z + 1 = 0$$

hyperbola

$$x = 1 + t = 2 - s$$
$$y = -1 + 4t = 3 + s$$
$$z = 2t = 1 + 3s$$

solve xy $t = 1 - s$

$t = 1, \quad s = 0$

but then

$z = 2 \neq 1 + 3(0)$

so SKEW

point $(0,0,0)$

$$\vec{n} = \langle 1, 4, 2 \rangle \times \langle -1, 1, 3 \rangle$$

$$\langle 10, -5, 5 \rangle$$

plane $= 10x - 5y + 5z = 0$