

KEY

Math 225: Exam the Second

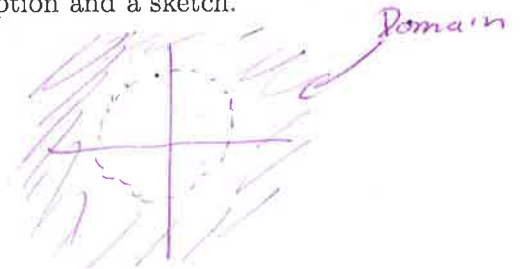
You have two hours to take this closed-book, closed-note, and closed-colleague exam. You may use a basic calculator for arithmetic, trig functions, logarithms and exponentials, but no graphing or calculus functions.

1. Let $f(x, y) = \frac{1}{\sqrt{x^2+y^2-1}}$

(a) What is the domain of f ? Give both an algebraic description and a sketch.

Denominator cannot = 0.
Radical term cannot be negative

$\Rightarrow x^2+y^2-1 > 0, x^2+y^2 > 1$

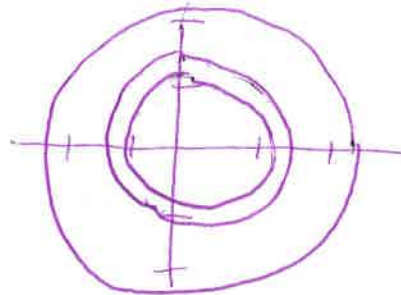


(b) Draw level curves for f for values of $\frac{1}{2}, 1, 2$

$k = \frac{1}{\sqrt{x^2+y^2-1}} \Rightarrow x^2+y^2 = 1 + \frac{1}{k^2}$

level curves are circles of radius $\sqrt{1 + \frac{1}{k^2}}$

k	$\frac{1}{2}$	1	2	$\rightarrow \infty$
radius	$\sqrt{5}$	$\sqrt{2}$	$\sqrt{\frac{5}{4}}$	$\rightarrow 1$



(c) Are there level curves for $f \leq 0$? Explain.

no. $\sqrt{x^2+y^2-1} > 0 \Rightarrow$ all values ^{in the range,} are positive

2. Let $f(x, y) = x^2 + y^3 + 4y + x \cos(y)$

(a) Find the tangent plane at the point $(3, 0)$ and use it to approximate $f(\pi, 0.1)$

$$f(3, 0) = 9 + 0 + 0 + 3 \cos 0 = 12$$

$$f_x = 2x + \cos y \xrightarrow{(3, 0)} 6 + 1 = 7$$

$$f_y = 3y^2 + 4 + x \sin y \xrightarrow{(3, 0)} 0 + 4 + 0 = 4$$

$$T_{\text{plane}}: z - 12 = 7(x - 3) + 4(y - 0)$$

$$f(\pi, 0.1) \approx$$

$$12 + 7(3.14 - 3) + 4(0.1 - 0)$$

$$= 12 + 7(0.14) + 4(0.1)$$

$$= \underline{13.38}$$

(b) Find $D_{\vec{u}}f$ at $(3, 0)$ as we move towards $(2, 1)$.

$$(3, 0) \rightarrow (2, 1) \xrightarrow{\vec{v}} \langle -1, 1 \rangle \xrightarrow{\text{unit}} \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$D_{\vec{u}}f = \vec{\nabla}f \cdot \vec{u} = \langle 7, 4 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left(-\frac{3}{\sqrt{2}} \right)$$

3. (a) Find the equation to the tangent plane to the surface $x^3y + y^3z + z^3x = 5$ at the point $(2, 1, -1)$.

$$\vec{\nabla}F = \langle 3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3 \rangle \Big|_{(2, 1, -1)} = \langle 11, 5, 7 \rangle$$

$$T_{\text{plane}}: 11(x - 2) + 5(y - 1) - 7(z + 1) = 0$$

(b) Find $\frac{\partial z}{\partial y}$ at this point.

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \underline{\underline{-\frac{5}{7}}}$$

4. Let $f(x, y) = x^2 - 2x + 2y^2 - 4y + 5$.

(a) Find and classify the only critical point of f .

$$f_x = 2x - 2 = 0$$

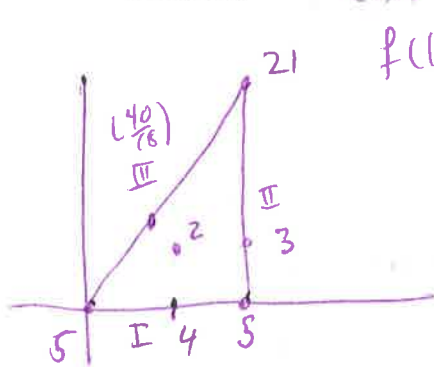
$$f_y = 4y - 4 = 0$$

c.p. = (1, 1)

$$D = (2)(4) - (0)^2 = 8 > 0 \text{ so } (1, 1) \text{ is a maximum}$$

Since $f_{xx} > 0 \rightarrow$ graph is concave up \rightarrow min

(b) Find the maximum and minimum value of f on the triangle with vertices (0, 0), (2, 4) and (2, 0). (1, 1) a candidate



$$f(1, 1) = 1 - 2 + 2 - 4 + 5 = 2$$

I: $y = 0$
opt $f = x^2 - 2x + 5$

$$f' = 2x - 2 \rightarrow x = 1$$

$$f = 1 - 2 + 5 = 4$$

$$f(0) = 5$$

$$f(4) = 5$$

II $x = 2$

$$f = 2y^2 - 4y + 5$$

$$f' = 4y - 4 = 0 \quad y = 1$$

$$f(1) = 2 - 4 + 5 = 3$$

$$f(0) = 5$$

$$f(4) = 32 - 16 + 5 = 21$$

III $y = 2x$
 $f = x^2 - 2x + 8x^2 - 8x + 5 = 9x^2 - 10x + 5$

$$f' = 18x - 10 = 0 \quad x = \frac{10}{18}$$

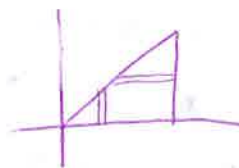
$$f\left(\frac{10}{18}\right) = \frac{900}{18^2} - \frac{100}{18} + 5$$

$$\frac{900}{18^2} - \frac{1800}{18^2} + \frac{1620}{18^2} = \frac{720}{18^2} = \frac{40}{18}$$

5. Calculate

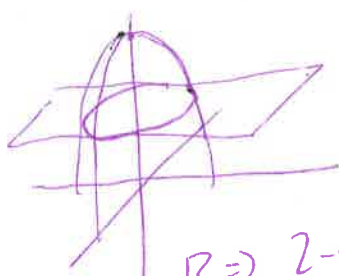
$$\int_0^3 \int_y^3 e^{x^2} dx dy$$

by first reversing the order of integration.



$$\begin{aligned} \int_0^3 \int_0^x e^{x^2} dy dx &= \int_0^3 x e^{x^2} dx \\ &= \left[\frac{e^{x^2}}{2} \right]_0^3 \\ &= \left[\frac{e^9 - 1}{2} \right] \end{aligned}$$

6. Find the volume bound by the paraboloid $z = 2 - x^2 - y^2$ and $z = 1$.



$R \Rightarrow z = 2 - x^2 - y^2 = 1$
 $x^2 + y^2 = 1$
 unit circle

$$\iint_R (2 - x^2 - y^2) - 1 dA$$

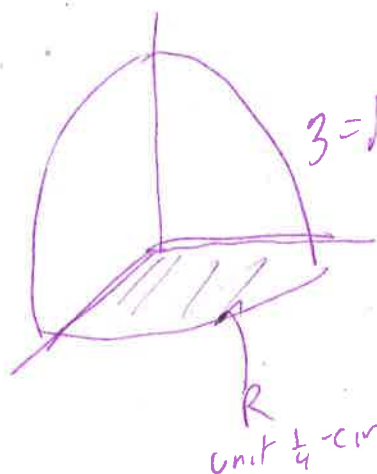
$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3 dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} d\theta = \boxed{\pi}$$

7. Find the volume of the unit sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant using calculus (You may check your answer if you remember the formula for the volume of a sphere!).



$$\begin{aligned}
 & \int_R \sqrt{1-x^2-y^2} \, dA \\
 &= \int_0^{\pi/2} \int_0^1 (\sqrt{1-r^2}) r \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left[-\frac{2}{3}(1-r^2)^{3/2} \right]_0^1 d\theta \\
 &= \int_0^{\pi/2} \frac{2}{3} d\theta = \frac{\pi}{6}
 \end{aligned}$$

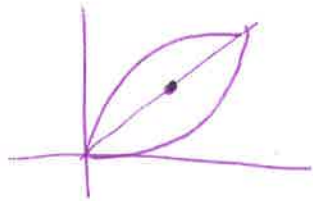
8. Set up, but don't compute, the integral for the surface area of $f(x, y) = \sqrt{x^2 + y^2 - 1}$ over $[3, 4] \times [4, 6]$.

$$f_x = \frac{1}{2}(x^2 + y^2 - 1)^{-1/2} (2x) \quad (f_x)^2$$

$$f_y = \frac{1}{2}(x^2 + y^2 - 1)^{-1/2} (2y) \quad (f_y)^2$$

$$S.A. = \int_3^4 \int_4^6 \sqrt{1 + \frac{x^2}{x^2 + y^2 - 1} + \frac{y^2}{x^2 + y^2 - 1}} \, dy \, dx.$$

9. Find the center of mass of a plate in the shape of the area between the curves $y = x^2$ and $x = y^2$ in the first octant, if the density of the plate is given by the function $\rho(x, y) = xy$. (Hints: You may save yourself some work by employing symmetry. The mass of this plate is $\frac{1}{12}$).



NB: $\bar{x} = \bar{y}$

$$\bar{x} = \frac{1}{\text{mass}} \iint_R x \rho(x, y) \, dA = 12 \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy \, dx$$

$$= 12 \int_0^1 \left. \frac{x^2 y^2}{2} \right|_{x^2}^{\sqrt{x}} dx$$

$$= 12 \int_0^1 \left(\frac{x^3}{2} - \frac{x^6}{2} \right) dx = 6 \left[\frac{x^4}{4} - \frac{x^7}{7} \right]_0^1$$

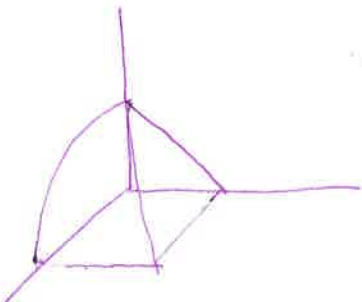
$$= \frac{6 \cdot 3}{28} = \frac{9}{14}$$

C.O.M = $\left(\frac{9}{14}, \frac{9}{14} \right)$

10. Set up the bounds for

$$\iiint_E f(x, y, z) \, dV,$$

where E is bound by $x = 0, y = 0, z = 0, x = 1 - z^2$, and $y + z = 1$.



$$\int_0^1 \int_0^{1-z} \int_0^{1-z^2} f(x, y, z) \, dx \, dy \, dz$$