Final Review May 12, 2015

1. $\vec{a} = \langle 1,2,4\rangle$ $\vec{b} = \langle 0,3,6\rangle$ $\vec{r} = \langle x,y,z\rangle$

$(\vec{r}-\vec{a})\cdot\langle \vec{r}-\vec{b}\rangle = 0$

$\langle x-1, y-2, z-4\rangle \cdot \langle x, y-3, z-6\rangle = 0$

$x^2 - x + y^2 - 5y + 6 + z^2 - 10z + 24 = 0$

$x^2 - x + y^2 - 5y + z^2 - 10z = -30$

$x^2 - x + \frac{1}{4} + y^2 - 5y + \frac{25}{4} + z^2 - 10z + 25 = -30 + \frac{1}{4} + \frac{25}{4} + 25$ radius $= \frac{\sqrt{6}}{2}$ ← half distance btwn points

$(x-\frac{1}{2})^2 + (y-\frac{5}{2})^2 + (z-5)^2 = \frac{3}{2}$ center: $(\frac{1}{2}, \frac{5}{2}, 5)$ ← midpoint

2a. Plane with $(0,0,2)$, $(1,2,3)$ $(-1,4,5)$

$$\begin{array}{r} \langle 1,2,1\rangle \\ \times \langle -1,4,3\rangle \\ \hline \langle 2,-4,6\rangle \end{array}$$

$2x - 4y + 6(z-2) = 0$

$\boxed{2x - 4y + 6z = 12}$

b. Does $\vec{l}(t) = \langle 1+3t, 5, 2-t\rangle$ intersect plane? Where or why not?

$2(1+3t) - 4(5) + 6(2-t) = 12$

$2 + 6t - 20 + 12 - 6t = 12$ $-18 = 0$ WUT

$\vec{v} = \langle 3,0,-1\rangle \perp \vec{n} = \langle 2,-4,6\rangle$ line is $\perp$ to normal of plane, so line is parallel to plane don't intersect

3. $\vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{\cos^2 t}\rangle$

a) Range of values for each component: $-1 \le x \le 1$ $-1 \le y \le 1$ $0 \le z \le 1$

b) What two surfaces intersect? x y z

$x^2 + y^2 = 1$ cylinder → $\cos^2 t + \sin^2 t = 1$ ✓

$z = x^2$ parabolic cylinder $\cos^2 t = \cos^2 t$ ✓

c) Tangent to curve at $t = \frac{\pi}{4}$

$\vec{r}(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2})$ $\vec{r}\,'(t) = \langle -\sin t, \cos t, -2\cos t \sin t\rangle$

$\vec{r}\,'(\frac{\pi}{4}) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1)$ Tangent line: $\vec{l}(t) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\rangle + t\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\rangle$

4. $f(x,y) = xy + x^2 - e^y$

a) Find tangent plane at $(3,0)$

$f_x = y + 2x$ $f_x(3,0) = 6$ $f(3,0) = 8$ $z = 8 + 6(x-3) + 2y$

$f_y = x - e^y$ $f_y(3,0) = 2$ equivalent

or $\vec{n} = \langle -f_x, -f_y, 1\rangle$

$-6(x-3) - 2(y-0) + 1(z-8) = 0$

b. $D_{\vec{u}} f$ moving from $(3,0)$ to $(4,2)$ $\quad \vec{u} = \langle 1,2 \rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$\nabla f \cdot \vec{u} = \langle 6, 2 \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \frac{10}{\sqrt{5}}$ ← rate of increase in that direction

f_x, f_y at $(3,0)$

5. Minimize $f(x,y,z) = 3x + 2y + z$ $\quad (x,y,z>0)$ st $xyz = 100$

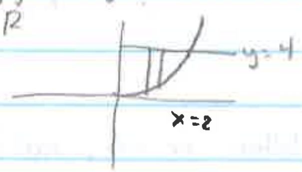
$3 = \lambda yz \longrightarrow 3x = \lambda xyz$ $\qquad 3x = 2y = z$

$2 = \lambda xz \longrightarrow 2y = \lambda xyz$ $\qquad (\frac{z}{3})(\frac{z}{2})z = 100$ $\qquad x = \frac{\sqrt[3]{600}}{3}$

$1 = \lambda xy \longrightarrow z = \lambda xyz$ $\qquad \frac{z^3}{6} = 100$ $\quad z^3 = 600$ $\quad z = \sqrt[3]{600}$

$\boxed{\text{Minimum} = 3\sqrt[3]{600}}$ $\qquad y = \frac{\sqrt[3]{600}}{2}$

6. $\iint_R x+y \, dA$ where R is bound by $x=0$ $y=4$ and $y=x^2$ $\quad dy\,dx$ and $dx\,dy$



$0 \le x \le 2$
$x^2 \le y \le 4$

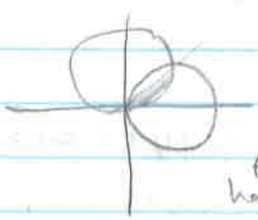
$\int_0^2 \int_{x^2}^4 x+y \, dy\,dx = \int_0^2 xy + \frac{y^2}{2} \Big|_{x^2}^4 dx$

$= \int_0^2 4x + 8 - x^3 - \frac{1}{2}x^4 \, dx = 2x^2 + 8x - \frac{x^4}{4} - \frac{x^5}{10} \Big|_0^2$

or $\qquad = 8 + 16 - 4 - \frac{32}{10} = 16.8$

$0 \le x \le \sqrt{y}$
$0 \le y \le 4$ and integrate $dx\,dy$

7. Area contained in $x^2+y^2 = 2x$ and $x^2+y^2=2y$



$r = 2\cos\theta$ $\qquad r = 2\sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\qquad 0 \le \theta \le \pi$

intersection of two curves

find half of it and Double → 2 $\qquad 2\int_0^{\frac{\pi}{4}} \int_0^{2\sin\theta} r \, dr\,d\theta$

$= 2\int_0^{\frac{\pi}{4}} \frac{(2\sin\theta)^2}{2} d\theta = 4\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta = 2\int_0^{\frac{\pi}{4}} 1 - \cos 2\theta \, d\theta = 2\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\frac{\pi}{4}}$

$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$

8. $\iiint_E 9 - x^2 - y^2 \, dV$ $\quad$ E is upper hemisphere radius 3

$\leftarrow x^2+y^2+z^2 = 9$ $\qquad$ cylindrical $\qquad \int_0^{2\pi}\int_0^3\int_0^{\sqrt{9-r^2}} (9-r^2)\, r\,dz\,dr\,d\theta$

$z = \sqrt{9-x^2-y^2}$

$0 \le z \le \sqrt{9-r^2}$

or spherical

r is $\rho\sin\phi$ in spherical

$0 \le \rho \le 3$
$0 \le \theta \le 2\pi$ $\qquad \int_0^3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (9 - \rho^2 \sin^2\phi)\, \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$
$0 \le \phi \le \frac{\pi}{2}$

$$\int_0^{2\pi}\int_0^3 (9-r^2)rz\Big|_0^{\sqrt{9-r^2}}\,dr\,d\theta$$

cylindrical one: $\displaystyle\int_0^{2\pi}\int_0^3 (9-r^2)^{\frac{3}{2}}\,r\,dr\,d\theta = \int_0^{2\pi} -\frac{2}{5}\cdot\frac{1}{2}(9-r^2)^{\frac{5}{2}}\Big|_0^3 d\theta = \int_0^{2\pi} \frac{243}{5}\,d\theta$

$= \frac{2}{5}(243\pi)$

$= \frac{486\pi}{5}$

9. $\displaystyle\int_C 2xy\,dx + (x^2+y^2)\,dy \quad C: (1,1)\to(4,5)$

conserv? $P_y = 2x \quad Q_x = 2x$ ✓ not closed can use FTC I

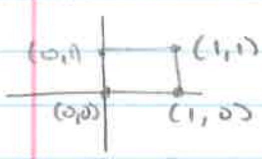
$f_x = 2xy$

$f = x^2y + g(y)$

$f_y = x^2 + g'(y) = x^2 + y^2 \quad g'(y) = y^2 \quad g(y) = \frac{y^3}{3} \qquad f = x^2y + \frac{1}{3}y^3\Big|_{(1,1)}^{(4,5)}$

f at endpoints $= \left(80 + \frac{125}{3}\right) - \left(1 + \frac{1}{3}\right) = \frac{361}{3}$

10. $\displaystyle\oint_C 3y\,dx + x^2\,dy$ not conserv, use Green's



$$= \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_0^1\int_0^1 2x-3\,dy\,dx = \int_0^1 2x-3\,dx$$

$$= x^2 - 3x\Big|_0^1 = -2$$

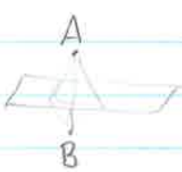
11. $\displaystyle\oint \overset{P}{x^2}\,dx + \overset{Q}{\cos(y)}\,dy + \overset{R}{e^z}\,dz \qquad C: x^2+y^2=1$ and $z = x^2 - y^2$

closed and conservative so 0

$P_y = 0 \quad Q_x = 0$ ✓ $\quad R_x = 0 \quad P_z = 0$ ✓ $\quad Q_z = 0 \quad R_y = 0$ ✓

12. Points equidistant from $A=(2,3,4)$ $B=(6,-1,10)$ find $P=(x,y,z)$

such that $d(P,A) = d(P,B)$



$d(P,A) = \sqrt{(x-2)^2 + (y-3)^2 + (z-4)^2} = \sqrt{(x-6)^2 + (y+1)^2 + (z-10)^2}\;\; d(P,B)$

$x^2 - 4x + 4 + y^2 - 6y + 9 + z^2 - 8z + 16 = x^2 - 12x + 36 + y^2 + 2y + 1 + z^2 - 20z + 100$

$8x - 8y + 12z = 108$

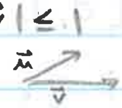
or $\quad 4x - 4y + 6z = 54 \qquad \vec{n} = \overrightarrow{AB}$

13. $\vec{u}$ and $\vec{v}$ unit vectors in $\mathbb{R}^3$

a. $|\vec{u}\times\vec{v}|$ max/min value $\qquad |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta \longleftarrow$ cross product vector is longest when $\vec{u} \perp \vec{v}$

$0 \le |\vec{u}\times\vec{v}| \le 1$

also $|\vec{u}\times\vec{v}| =$ area of parallelogram



14. $\vec{r}(t) = \langle t, 2t, t^3 \rangle$

a) Tangent when $t=1$ $\quad \vec{r}(1) = \langle 1, 2, 1 \rangle$

$\vec{r}'(t) = \langle 1, 2, 3t^2 \rangle \big|_{t=1} = \langle 1, 2, 3 \rangle \qquad \vec{\ell}(t) = \langle 1, 2, 1 \rangle + t\langle 1, 2, 3 \rangle$

b) Find speed and acceleration when $t=1$

speed $= |\vec{r}'(1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \qquad \vec{T}(1) = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$

acceleration $= \vec{r}''(t) = \langle 0, 0, 6t \rangle \big|_{t=1} = \langle 0, 0, 6 \rangle$

$\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ ← unit velocity

15. $f(x,y) = x^2 + 2xy + \ln y$

a) Find tangent plane at $(2,1)$

$f_x = 2x + 2y \quad f_x(2,1) = 6 \quad f(2,1) = 4+4+0 \qquad z - 8 = 6(x-2) + 8(y-1)$

$f_y = 2x + \frac{1}{y} \quad f_y(2,1) = 5 \qquad = 8$

b) $D_{\vec{u}}f \quad (2,1) \to (0,0) \quad \vec{u} = \langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle$

$\vec{\nabla} f \cdot \vec{u} = \langle 6, 5 \rangle \cdot \langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle = \frac{-17}{\sqrt{5}}$

16. Find and classify critical points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$

$f_x = 3 - 3x^2 = 0 \qquad f_y = -4y + 4y^3 = 0$

$3 = 3x^2 \qquad 4y^3 = 4y$

$x = \mp 1 \qquad y = \mp 1, 0$

$(1,1) \quad (1,0) \quad (1,-1) \qquad D: f_{xx} \cdot f_{yy} - (f_{xy})^2$

$(-1,1) \quad (-1,0) \quad (-1,-1)$

$f_{xx} = -6x \quad f_{yy} = -4 + 12y^2 \quad f_{xy} = 0 \quad D = (-6x)(-4+12y^2)$

$D_{(1,1)} = (-6)(-4+12) < 0$ saddle

$D_{(1,0)} = (-6)(-4) > 0 \quad f_{xx} < 0$ so max

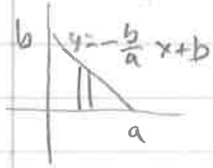
$D_{(1,-1)} = (-6)(8) < 0$ saddle

$D_{(-1,1)} = (6)(-4+12) > 0$ min

$D_{(-1,0)} = (6)(-4) < 0$ saddle

$D_{(-1,-1)} = (6)(8) > 0$ min

17. $(0,0) \quad (a,0) \quad (0,b)$

a) 

$\int_0^a -\frac{b}{a}x + b \, dx = \frac{1}{2}ab$

b. $(0,0,0)$
$(a,0,0)$
$(0,b,0)$
$(0,0,c)$ $\quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\int_0^a \int_0^{-\frac{b}{a}x+b} c - \frac{c}{a}x - \frac{c}{b}y \, dy \, dx = \frac{1}{3}(\text{base})(\text{height}) = \frac{1}{6}abc$$

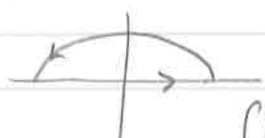
c. Going up one dimension, volume might be $\frac{1}{24}abcd$

18. $\int_C \overset{P}{(y\cos x)}\,dx + \overset{Q}{(y+\sin(x))}\,dy$ $\quad C$: any path from $(0,1)$ to $(\frac{\pi}{2},1)$

$P_y = \cos x \quad Q_x = \cos x$ ✓ $\longrightarrow$ conservative = path independent

$$f = y\sin x + \frac{y^2}{2}\Big|_{(0,1)}^{(\frac{\pi}{2},1)} = \left(\sin\frac{\pi}{2} + \frac{1}{2}\right) - \left(0 + \frac{1}{2}\right) = \sin\frac{\pi}{2} = 1$$

19. $\oint_C (x^2+2y)\,dx + (3x+\sqrt{y})\,dy$ $\quad C$: starts at $(1,0)$, counterclockwise along top half unit circle, then back

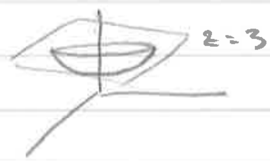


Green's Theorem!

$$\iint_A 3-2 \, dA = \iint 1 \, dA = \text{area of } A = \frac{\pi}{2}$$

20. $z^2 = 1+x^2+y^2$

a) Hyperboloid of two sheets



b) Volume bound by $z=3$ in first octant

$$\iint_R 3-\sqrt{1+x^2+y^2} \, dA \qquad R \text{ is intersection} \qquad x^2+y^2+1 = 3^2, \quad x^2+y^2 = 8$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} (3-\sqrt{1+r^2})\,r \, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} 3r - r(1+r^2)^{\frac{1}{2}} \, dr\, d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{2}r^2 - \frac{1}{3}(1+r^2)^{\frac{3}{2}} \Big|_0^{2\sqrt{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} 12 - \left(\frac{1}{3}\right)(9)^{\frac{3}{2}} + \frac{1}{3} \, d\theta = \int_0^{\frac{\pi}{2}} \frac{10}{3} \, d\theta = \frac{10\pi}{6}$$