KEY

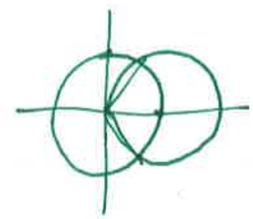
Math 225: Exam The First

February 24, 2017

You have two hours to complete this exam. You may use a calculator for arithmetic, trigonometric, exponential, and logarithm functions only (ie, no graphing and no calculus functions). Please read all directions carefully.

1. (a) Find the points (in *rectangular coordinates*) where the graphs of $r = 4\cos(\theta)$ and $r = 2$ intersect.

$4\cos(\theta) = 2 \quad \cos\theta = \frac{1}{2} \quad \theta = \pi/3$



points $= (2\cos \pi/3, \pm 2\sin \pi/3)$

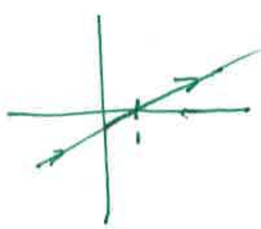
$= (1, \sqrt{3})$

AND $\hookrightarrow (1, -\sqrt{3})$

(b) Set up, but don't evaluate, the integral (in polar coordinates) to find the area outside of $r = 2$, but inside $r = 4\cos(\theta)$.

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} (4\cos\theta)^2 - (2^2)\, d\theta$$

2. (a) Eliminate the parameter to graph the parametric curve $x = 2t + 1, y = t$.



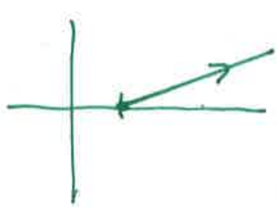
$y = t \to x = 2y + 1$

$y = \frac{1}{2}(x - 1)$

$y = \frac{x}{2} - \frac{1}{2}$

(b) Use your graph from part (a) to graph the parametric curve for $x = t^2 + 1, y = \dfrac{t^2}{2}$.

$x \geq 1 \qquad y \geq 0$



3. Consider the parametric equations

$$x = t^2 - t, y = 2t - t^2$$

(a) At what point does this curve have a vertical tangent?

Vert tan $\rightarrow \frac{dx}{dt} = 0 \quad 2t-1=0 \rightarrow t = 1/2$

$x = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$ point $\left(\frac{-1}{4}, \frac{3}{4}\right)$

$y = 2\left(\frac{1}{2}\right) - \frac{1}{4} = 3/4$

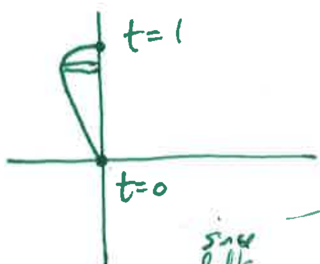
(b) For which values of t is $x \leq 0$ and $y \geq 0$?

$t^2 - t \leq 0$
$\Rightarrow t^2 \leq t$
$0 \leq t \leq 1$

$y \geq 0$
$2t - t^2 \geq 0$
$2t \geq t^2$
$2 \geq t \geq 0$

so Both happen for $0 \leq t \leq 1$

(c) Use these values to find the area bound by the curve and the y-axis in the *second* quadrant.



since left of axis

$\int_0^1 x \, dy$

$= -\int_0^1 (t^2 - t)(2-2t) \, dt$

$= -\int_0^1 -2t^3 + 2t^2 + 2t^2 - 2t \, dt$

$= \int_0^1 2t^3 - 4t^2 + 2t \, dt$

$= \frac{2t^4}{4} - \frac{4}{3}t^3 + t^2 \Big]_0^1 = \frac{1}{2} - \frac{1}{3} + 1 = 7/6$

(d) Give an equation of a hyperboloid of two sheets that has symmetry about the y-axis which goes through the points $(0, 3, 0)$ and $(0, -3, 0)$.

Two sheets $\rightarrow$ y axis

$y^2 = x^2 + z^2 + k$

If $x = z = 0$
$y = 3$
$\Rightarrow k = 9$

$y^2 = x^2 + z^2 + 9$

4. Let $A = (3,1,2)$, $B = (7,3,6)$ and $P = (x,y,z)$.

(a) Find the distance from A to B.

$$d = \sqrt{(7-3)^2+(3-1)^2+(6-2)^2}$$

$$= \sqrt{4^2+2^2+4^2} = \sqrt{36} = 6$$

(b) Write an equation to express

$$\overrightarrow{AP} \perp \overrightarrow{BP}$$

$$\rightarrow \langle x-3, y-1, z-2\rangle \cdot \langle x-7, y-3, z-6\rangle = 0$$

$$(x-3)(x-7)+(y-1)(y-3)+(z-2)(z-6) = 0$$

(c) Simplify your equation to the formula for a sphere. What are the center and radius of the sphere relative to A and B?

$$x^2-10x+21+y^2-4y+3+z^2-8z+12 = 0$$

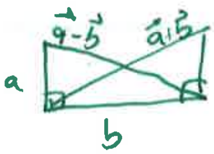
$$x^2-10x+25+y^2-4y+4+z^2-8z+16 = 4+1+4$$

$$(x-5)^2+(y-2)^2+(z-4)^2 = 9$$

$$C = (5,2,4) = \text{midpoint}(A,B)$$

$$r = 9 = \tfrac{1}{2}\,\text{dist}(A,B)$$

5. Suppose that $\mathbf{a} \perp \mathbf{b}$. Prove that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$. (Draw a picture if it helps).



$$|\vec a+\vec b|^2 = (\vec a+\vec b)\cdot(\vec a+\vec b) = \vec a\cdot\vec a + \cancel{\vec a\cdot\vec b}^{\,0} + \cancel{\vec b\cdot\vec a}^{\,0} + \vec b\cdot\vec b = \vec a\cdot\vec a + \vec b\cdot\vec b$$

$$|\vec a-\vec b|^2 = (\vec a-\vec b)\cdot(\vec a-\vec b) = \vec a\cdot\vec a - \vec a\cdot\vec b - \vec b\cdot\vec a + \vec b\cdot\vec b = \vec a\cdot\vec a + \vec b\cdot\vec b$$

These are equal

$$\Rightarrow |\vec a+\vec b| = |\vec a-\vec b|$$

6. Find the equation of the line through the point (2,1,4) that is perpendicular to the plane $4x - y + 3z = 7$.

point (2, 1, 4)
dir vector $\langle 4, -1, 3\rangle$

line
$$\vec\ell(t) = \langle 2,1,4\rangle + \langle 4,-1,3\rangle t$$

7. Find the plane of intersection of the lines $\langle 2+t, 4t, 1+3t\rangle$ and $\langle -1-2s, 6+s, 10+3s\rangle$.

plane of intersection
point (-1, 6, 10)

normal = $\langle 1, 4, 3\rangle \times \langle -2, 1, 3\rangle$

$$\langle 9, -9, 9\rangle$$

plane $9(x+1) - 9(y-6) + 9(z-10) = 0$

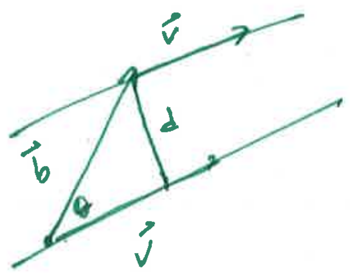
8. In this exercise, we develop the formula for the distance between two parallel lines.

(a) The figure on the board each with direction vector **v** starting at a point. We connect the starting points with a vector **b**. Also, we label the length that will be the distance d between the lines at the head of vector **b**.

(b) Consider the angle θ between **b** and **v**. We calculate $\sin(\theta)$ two ways...

i. ...as a ratio involving the distance between the lines.

ii. ...as a result of the cross product of **b** and **v**.



$$\sin\theta = \frac{d}{|\vec{b}|} = \frac{|\vec{b}\times\vec{v}|}{|\vec{b}||\vec{v}|}$$

(c) Now, set these calculations equal and simplify for an elegant formula for the distance d between the two lines.

$$\frac{d}{\cancel{|\vec{b}|}} = \frac{|\vec{b}\times\vec{v}|}{\cancel{|\vec{b}|}|\vec{v}|} \rightarrow d = \frac{|\vec{b}\times\vec{v}|}{|\vec{v}|}$$

(d) Use your formula to find the distance between $\langle 2+3t, 5t, -1+t\rangle$ and $\langle 1+3t, 4+5t, 7+t\rangle$.

$\vec{v} = \langle 3, 5, 1\rangle$

$$\begin{array}{r} \langle 3, 5, 1\rangle \\ \vec{b} = (2,0,-1) \text{ to } (1,4,7) \Rightarrow \langle -1, 4, 8\rangle \\ \hline \langle 36, -25, 17\rangle \end{array}$$

$$d = \frac{\sqrt{36^2+25^2+17^2}}{\sqrt{3^2+5^2+1^2}}$$

9. Consider the function $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$ $= (x^2+y^2-1)^{1/2}$

(a) Find f_x, f_y and f_{xy} for this function.

$f_x = \frac{1}{2}(x^2+y^2-1)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{x^2+y^2-1}}$

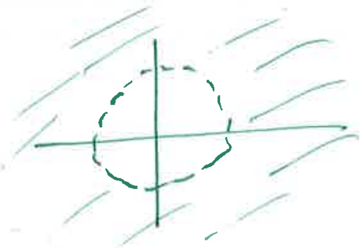
$f_y = \frac{1}{2}(x^2+y^2-1)^{-1/2} \cdot 2y = \dfrac{y}{\sqrt{x^2+y^2-1}}$

$f_{xy} = -\frac{\cancel{1}y}{\cancel{2}}(x^2+y^2-1)^{3/2} \cdot 2x = \dfrac{-xy}{(\sqrt{x^2+y^2-1})^3}$

(b) Find and sketch the domain for this function.

$x^2+y^2 > 1$

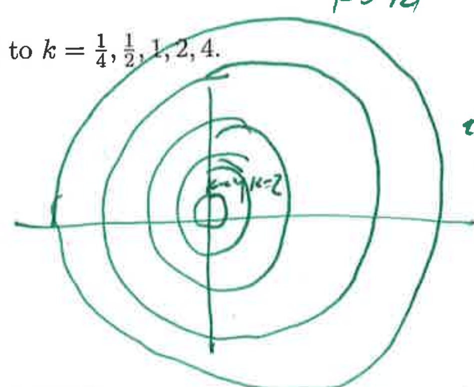
to be positive



$k = 1/4$

(c) Draw level curves for this surface corresponding to $k = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$.

$\dfrac{1}{\sqrt{x^2+y^2-1}} = k \Rightarrow x^2+y^2 = \frac{1}{k^2} + 1$



supposed to be larger

Volcano!

(d) What 'geologic' structure does the graph of f resemble?

10. (Bonus) Tell me about a panel or event that you attended in conjunction with yesterday's Power and Privilege Symposium.