KEY

Math 225: Exam the Second April ~~12~~ 17, 20~~15~~17

You have two hours to complete this closed-book, closed-note, closed colleague exam. You may use a calculator for arithmetic only (trig functions and exponentials are okay, but no plotting and no calculus).

1. (a) Find the equation of the tangent plane to the function $f(x, y) = \dfrac{x}{x+y}$ at the point (2,1).

$f(2,1) = \frac{2}{2+1} = 2/3$

$f_x = \frac{(x+y)(1)-(x)(1)}{(x+y)^2} = \frac{y}{(x+y)^2} = \frac{1}{9}$

$f_y = -x(x+y)^{-2} = \frac{-x}{(x+y)^2} = -\frac{2}{9}$

plane: $z = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1)$

(b) Find $D_{\mathbf{u}}(f)$ as we move from (2,1) in the direction of $\mathbf{v} = \langle 3, 4\rangle$. $\vec{u} \to \langle \frac{3}{5}, \frac{4}{5} \rangle$

$\vec{\nabla} f \cdot \vec{u} = \langle \frac{1}{9}, \frac{-2}{9} \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \left(\frac{3}{45} - \frac{8}{45}\right) = \frac{-5}{45} = \frac{-1}{9}.$

2. Find the maximum and minimum value of $x^2+y^2+z^2$ subject to the constraint $x+3y+4z = 10$.

$f = x^2+y^2+z^2$ Const. $x + 3y + 4z = 10 = g$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$2x = \lambda$

$2y = 3\lambda$

$2z = 4\lambda$

$x = \frac{\lambda}{2}$ $y = \frac{3\lambda}{2}$ $z = \frac{4\lambda}{2}$ $\rightarrow$ $\frac{\lambda}{2} + \frac{9\lambda}{2} + \frac{16\lambda}{2} = 10$

$26\lambda = 20$

$\lambda = \frac{20}{26} = 10/13$

$x = \frac{10}{26}$ $y = \frac{30}{26}$ $z = \frac{40}{26}$

This is a min, since f could be arbitrarily large (with negative x values).

$f = \frac{10^2+30^2+40^2}{26^2}$

3. Consider the function $f(x, y) = x^2 + kxy + y^2$, where k is a constant.

(a) Show that f has a critical point at $(0, 0)$ regardless of the choice of k.

$f_x = 2x + ky = 0$ @ $(0,0)$ both $f_x, f_y = 0$

$f_y = Kx + 2y = 0$

(b) For which values of k is $(0, 0)$ a local minimum?

$f_{xx} = 2 = f_{yy}$

$f_{xy} = k$

$D = 2 \cdot 2 - (k)^2 = 4 - k^2$

$4 - k^2 > 0 \Rightarrow 4 > k^2$ $\quad -2 \leq k \leq 2$

$f_{xx} > 0 \Rightarrow$ min

(c) For which values of k is $(0, 0)$ a local maximum?

$f_{xx} > 0 \Rightarrow$ no local max.

(d) For which values of k is $(0, 0)$ a saddle point?

$4 - k^2 \leq 0 \Rightarrow k^2 > 4 \Rightarrow K > 2$ or $K < -2$.

(e) For which values of k does the discriminant test require more investigation?

$4 - k^2 = 0 \Rightarrow K = \pm 2$

(f) Investigate f at these values of k and classify (hint: f factors nicely in these cases).

If $k = 2$

$f = x^2 + 2xy + y^2$

$= (x+y)^2 \geq 0$

If $K = -2$

$f = x^2 - 2xy + y^2$

$= (x-y)^2 \geq 0$

Since $f(0,0) = 0$ in both cases, it is a type of minimum

4. Find

$$\iint_R x\cos(3xy)\,dA$$

where $R = [0,1] \times [0,\pi]$.

for a clean u-sub do dy first

$$\int_0^1\int_0^\pi x\cos(3xy)\,dy\,dx = \frac{1}{3}\int_0^1\int_{y=0}^{y=\pi}\cos(u)\,du\,dx$$

$u = 3xy$

$du = 3x\,dx$

$$= \frac{1}{3}\int_0^1 \sin(3xy)\Big]_0^\pi dx$$

$$= \frac{1}{3}\int_0^1 \sin(3\pi x)\,dx$$

$$= \frac{-1}{3\pi}\,\frac{1}{3}\cos(3\pi x)\Big]_0^1$$

$$= -\frac{1}{9\pi}\left[\cos(3\pi) - \cos(0)\right]$$

$$= \frac{-1}{9\pi}\left[(-1) - 1\right] = 2/9\pi.$$

5. Find

$$\int_0^8\int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy.$$

flip the strip!

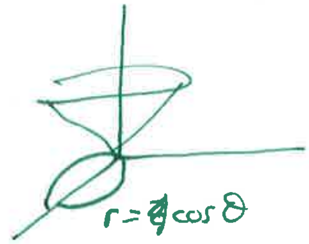
$$\int_0^2\int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^2 x^3 e^{x^4}\,dx = \frac{1}{4}\left[e^{x^4}\right]_0^2$$

$$= \frac{1}{4}\left[e^{16} - 1\right]$$

$r^2 = 4\cos\theta$

6. Find the volume below $z = \sqrt{x^2+y^2}$ and above $x^2+y^2 = 4x$.



$r = 4\cos\theta$

$-\pi/2 \le \theta \le \pi/2$

$$\int_{-\pi/2}^{\pi/2} \int_0^{4\cos\theta} r\,(r)\,dr\,d\theta$$

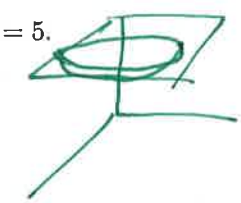
$$= \int_{-\pi/2}^{\pi/2} \frac{64\cos^3\theta}{3}\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{64}{3}\left[1-\sin^2\theta\right]\cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{64}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]d\theta = \frac{64}{3}\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]$$

$$= \frac{64}{3}\left(\frac{4}{3}\right) = \frac{256}{9}.$$

7. Find the volume bound by the upper sheet of $z^2 = x^2+y^2+9$ and $z = 5$.



$25 = x^2+y^2+9$

$x^2+y^2 = 16$

$$\iint_R \left(5-\sqrt{x^2+y^2+9}\right)dA$$

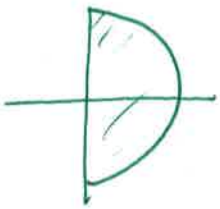
$$= \int_0^{2\pi}\int_0^4 5r - \left(\sqrt{r^2+9}\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{5r^2}{2} - \frac{1}{2}\cdot\frac{2}{3}(r^2+9)^{3/2}\Big|_0^4\,d\theta$$

$$= \int_0^{2\pi} \frac{80}{2} - \frac{1}{3}(125) + \frac{1}{3}(27)\,d\theta$$

$$= \int_0^{2\pi} 40 - \frac{98}{3}\,d\theta = \int_0^{2\pi} \frac{22}{3}\,d\theta = \frac{44\pi}{3}.$$

8. Find the center of mass of a thin plate in the shape of a semicircle of radius 1 along the y axis, which has density $\rho(x, y) = x^2$. Use symmetry where appropriate.



$$\text{MASS} = \iint_R x^2\,dA = \int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos^2\theta \, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\cos^2\theta \Big]_0^1 d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{4}\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{8}[1+\cos 2\theta]\,d\theta$$

$$= \frac{1}{8}\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= \pi/8$$

$$\bar{x} = \frac{1}{\text{mass}}\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^1 x\cdot x^2\,dA = \int_{-\pi/2}^{\pi/2}\int_0^1 r^5\cos^3\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{5}\cos^3\theta\,d\theta$$

$$= \frac{1}{5}\left[\frac{4}{3}\right] = \frac{4}{15}\cdot\frac{8}{\pi} = \frac{32}{15\pi}$$

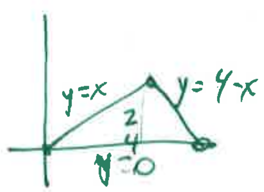
$\bar{y} = 0$ by symmetry

$$\text{C.o.m} = \left[\frac{32}{15\pi},\ 0\right]$$

9. Find the surface area of the plane $z = 3x + 6y + 4$ above the triangle with vertices $(0, 0)$, $(2, 2)$ and $(4, 0)$.

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\,dA = \iint_R \sqrt{1 + 9 + 36}\,dA$$

$$= \iint_R \sqrt{46}\,dA$$



$$= \sqrt{46}\ \text{Area}(R)$$

$$= \sqrt{46}\,(4) = 4\sqrt{46}$$

10. (Bonus) Tell me about a talk or poster that you heard/saw at the undergraduate conference yesterday.