Lab The Third
Pi and Buffon's Needles
Due Monday, February 29, at 5PM

In this lab, you will be numerically recreating Buffon's Monte Carlo Method for calculating π. This experiment from the early eighteenth century was the first example of *geometric probability.* In the experiment, a needle is dropped on a ruled surface, and we calculate the probability that it crosses one of the lines.

1 Probabilities-The Basic Problem

1.1 The Theoretical

For the first part of the lab, we are going to calculate a probability under specific conditions on the needle and the surface. We are assuming that the surface is ruled with horizontal lines 1 unit apart and that the needle is 1 unit long. We will denote by x the vertical distance of the center of the needle from the nearest line, and by θ the acute angle that the needle makes with the horizontal axis.

Question 1 *What is the permissible range of x and θ values? That is, find a, b, c and d such that, for any needle dropped, $a \leq x \leq b$ and $c \leq \theta \leq d$.*

We assume that the needle is equally likely to fall at any given place on the surface, thus the Probability Distribution Function should be a constant (see Stewart, Chapter 8 or 15 for a refresher on PDFs).

Problem 1 *Find the value k such that $f(x, \theta) = k$ is a probability distribution function, that is,*

$$\int_c^d \int_a^b k \; dx \; d\theta = 1$$

Problem 2 *Now, give conditions on x and θ under which the needle crosses one of the ruled lines. Plot this area on a set of axes with x as the variable dependent on θ.*

Problem 3 *Use your conditions from problem 2 to determine the probability p that a dropped needle crosses the line. That is, integrate the probability distribution function over the appropriate region from problem 2.*

1.2 The Applied

You will now simulate Buffon's experiment by generating a set of random values for x and θ and counting the number of needles that hit the lines in the experiments.

Question 2 *If we drop 100 needles on the sheet, how many would we* expect *to cross one of the lines? What about 1000? 1000000? n?*

You will now create two arrays and invoke a bit of programming within Maple to run the needle experiment on your data. You will need to invoke the package

```
with(RandomTools):
```

One method for generating a random array of data is to use the following command

```
A:=Generate(list(float(range=a..b, method=uniform), n));
```

The above command will generate an $n \times 1$ array named A, with entries that are floating decimals between a and b, uniformly distributed over that specified range. Warning: The above code will not work without first replacing a, b, and n with numbers.

Problem 4 *Write Maple code to generate an array of random of values for x and θ. Start with $n = 10$.*

You will now write a simple loop that will test each of your data points to see if the corresponding 'needle' 'crosses the line'. Use the following as a template.

```
k:=0 :
for i from 1 to n do
if Statement on X[i] and Θ[i] here then k := k + 1 end if:
end do;
k;
```

This code takes a counter, in this case, k, and initializes it to zero to begin. The code then opens a *for* statement that will run over your array of data. The code calls the individual points in the array by $A[i]$, where A is the name of your array. It then starts the following *if-then* loop: if the provided statement holds true (for your statement, you want the needle to cross the line), then it will bump the counter up by 1. At the end, we need to close down the *if-then* and *do* loops, and return the value of our counter.

Problem 5 *Determine the appropriate statement on $X[i]$ and $\Theta[i]$ to include in your code and run the code for your list. Does your answer seem 'about' right?*

Problem 6 *Rerun the experiment and increase n to 1000, 10000, and 100000. You may need to be patient in the last case, so make sure that the answers are seeming right before you dive in. If you haven't already done so, you might amend your worksheet so that Maple doesn't actually display the arrays for X and Θ. Create a table that compares your results to the expected results.*

2 A Different Needle

Now we wish to generalize the problem a little bit. Rather than a page ruled uniformly with intervals equal to the length of the needle, we suppose that the page is uniformly ruled by spaces of length d and the that the needle has length l.

Question 3 *Under this generalization of the problem, what are the new permissible values for x and θ, and what is the new probability distribution function?*

Problem 7 *Give a prose description of how the problem has now changed, and under what conditions on d and l the needle is more or less likely to cross a line.*

Problem 8 *Make your description mathematically rigorous, that is, give conditions on x and θ such that the needle will cross if the values x and θ are within those conditions. Your conditions will, of course, depend on d and l.*

Problem 9 *Make a new plot based on the generalized problem, similar to that given in problem 2.*

Question 4 *You should notice, based on your plot that there are two very different cases to consider, namely when $l < d$ and when $l > d$. What are the differences? Compute the probability in each case. Warning: One integral will be tricky. You may use Maple for this one.*

Problem 10 *Determine if your calculations hold up to the computation by fixing some values of l and d and rerunning the simulation.*

Problem 11 *Discuss the advantages and limitations of using this method as an approximation for π.*

Note: You are welcome to include historical background on the problem for an 'extra touch', but beware not to stumble upon the mathematics that are used to solve the problem as part of your searches. Other suggestions for 'extra touches' might include

- Investigating other applications of ‘geometric probability’
- Investigating how long Maple takes to work with large sets of numbers
- Investigating how Maple generates random numbers (and if they’re really ‘random’)