Lab The Third
Pi and Euler
Monday, March 9, 2009

In this lab, you will be exploring several proofs of Euler's Formula for the sum of the reciprocals of the squares.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1}$$

This series is known to converge, as it is a p-series (with $p = 2 > 1$), but the value to which it converges is fascinating. As we so often do in mathematics, we'll begin by reducing our initial problem to a 'simpler' one.

Problem 1 *Show that if we restrict Formula (1) to the even squares, that we get one-quarter of the desired value. Conclude that the sum of the reciprocals of the odd squares is* $\frac{\pi^2}{8}$*.*

Thus, our tack may be to compute reciprocals of odd squares only, or of all squares.

1 A Series Approach

This approach uses Taylor's formula to approximate a function with a polynomial. We begin with a light warmup...

Question 1 *What is the polynomial approximation to* $f(x) = \frac{1}{\sqrt{1-x}}$ *near* $x = 0$*? For which values of* x *does it converge? You'll want to list the first few terms, as well as determine a general formula for the* n*th term.*

Question 2 *How can we amend our formula to one for* $f(x) = \frac{1}{\sqrt{1-x^2}}$*, and in turn, for a formula for* $f(x) = \arcsin(x)$*.*

Now, a bit of mathematical trickery. Nothing up our sleeves ...

Problem 2 *Invert your formula from Question 2 by setting* $x = \sin(t)$*. What you should have is a really really complicated formula for* t*.*

Problem 3 *Now, integrate both sides of your formula from* $t = 0$ *to* $t = \frac{\pi}{2}$*. You may wish to use Maple to integrate the odd powers of* $\sin(t)$*, at least enough to determine a formula.*

If all of your coefficients are correct and have behaved up to this point, you should have your first formula for the sum of the reciprocals of the odd squares.

2 An Integration Approach

We now use a bit of multivariable calculus and a clever change of variables to prove the formula. You might want to look back at your Calc III notes on the Jacobian.

Question 3 *Determine*

$$\int_0^1 \int_0^1 x^{n-1} y^{n-1}\, dx\, dy$$

and explain what it might have to do with this problem.

Problem 4 *Use this result to express Formula (1) as an infinite sum of integrals. Write this integral in a closed form (hint: it is a geometric series). Why might the integral be considered 'improper'?*

The integral you've determined doesn't succumb easily to any familiar formulas or techniques, so we'll have to make the following substitution.

$$(x, y) \to (u - v, u + v)$$

Problem 5 *Determine the inverse transform and use it to rewrite your integral in terms of u and v. Be sure to take all of the relevant steps!!*

Your new region of integration should be symmetric about the u axis, as should your integrand, thus you may work strictly above the u-axis and double your results.

Problem 6 *Evaluate your integral, integrating first with respect to v and then with respect to u. The algebra and trig get a bit hairy with this second integral, so you may rely on Maple for that (though a full explanation of what is going on would merit 'Extra Touch' points).*

3 Euler's original approach

Euler was not blessed with the most recent version of the Maple software when he first discovered this formula, so we will now work through his original argument. We make use of the following algebraic fact.

> If a polynomial $p(x)$ has roots $r_1, r_2, \ldots, r_n$, then we can write $p(x) = c(x - r_1)(x - r_2)\ldots(x - r_n)$, for some constant c.

For example, if a quadratic polynomial $p(x)$ has roots -1 and 3, we can write $p(x) = c(x+1)(x-3)$. We don't know about the constant c without more information. You might justify this fact to yourself before going on. We can use this fact in conjunction with Taylor's formula to write a function as a polynomial in two different ways.

Problem 7 *Write* $f(x) = \sin(x)$ *as a polynomial in two different ways.*

- *Use Taylor's formula to write* $\sin(x)$ *as an infinite sum.*
- *List the roots of* $\sin(x)$ *to write it as an infinite product. Clean up your infinite product by combining those terms for which the roots have the same absolute value.*

These two polynomials should be the same, but they're not quite yet. For example, the coefficient of x in the first is 1, while the coefficient of x in the second is an infinite (and non-convergent![1]) product. We rectify this here.

Problem 8 *Rewrite your infinite product in the form*

$$\sin(x) = x \prod (1 - \frac{x}{r_i})$$

where r_i *are the roots of your polynomial. Perform the same cleanup as you did above, and make certain to index your product properly.*

Now your polynomials agree in their coefficient on x, as well as all other coefficients.

Problem 9 *Compare the coefficients on* x^3 *for each polynomial to, once again, prove Formula (1).*

Euler used this expansion to prove such other interesting facts as

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}$$

and, even more impressively (are we *sure* he didn't have Maple?)....

$$\sum_{k=1}^{\infty} \frac{1}{k^{26}} = \frac{2^{24}76977927\pi^{26}}{27!}$$

For all the marbles, you might take a pass at the first couple of those. In any event, research and report a bit about Euler's fascinating life.

[1]For you analysis diehards: Yes, we are sweeping a few of the convergence issues under the rug. (So did Euler) When you take Math 455, you'll see more of why this actually works