Polynomial Problems

1. What is the largest integer n such that $n+10$ divides n^3+100?

2. (Putnam 2005 B-1) Find a nonzero polynomial $P(x,y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

3. (Putnam 1963 B-2) Find all integers n such that $x^2 - x + n$ divides $x^{13} + x + 90$.

4. Prove that if f(x) is a polynomial with integral coefficients, and there exists a positive integer k such that none of the integers f(1),f(2),...,f(k) is divisible by k, then f(x) has no integral root.

5. Suppose that a and b are different roots of the polynomial $x^3 + x - 1$. Prove that ab is a root of $x^3 - x^2 - 1$.

6. (Putnam 2003 B-1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

7. (Putnam 1992 A-3) Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x=0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

8. (Putnam 1999 A-2) Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$