Counting Problems

1. (a) Suppose that you have a chessboard with two opposite corner squares removed. Can you tile the remaining squares with 2×1 dominoes?
   (b) Suppose that you have a chessboard with one square removed from the corner. Can you tile the remaining squares with 3×1 dominoes?
   (c) Suppose that you have a rectangular board $m \times n$ that you want to tile with $r \times 1$ dominoes. Prove that r divides either m or n (or both).

2. Prove that if, in a standard deck of 52 playing cards, there are more red cards in the top 26 than there are black cards in the bottom 26, then somewhere in the deck there are three consecutive cards of the same color.

3. (1994 Putnam B1) Find all positive integers n that are within 250 of exactly 15 perfect squares.

4. (1996 Putnam B1) Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

5. (1995 Putnam A4) Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k - 1 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

6. (1995 Putnam B1) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]