Calculus Problems

1. (1996 Putnam A1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

2. (1990 Putnam B1) Find all real-valued continuously differentiable functions f on the real line such that for all x,
$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

3. (1988 Putnam A2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

4. (1987 Putnam B1) Evaluate
$$\int_2^4 \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

5. Let f be a real valued function such that

   (a) f is increasing on [0,1].

   (b) $f(0) = 0$

   (c) f' exists and is increasing on $(0, 1)$.

   Prove that $g(x) = f(x)/x$ is increasing on $(0, 1)$.

6. (1983 Putnam A2) A clock's minute hand has length 4 and its hour hand length 3. What is the distance between the tips at the moment when it is increasing most rapidly?