Parity Problems

1. Let (a, b, c) be a Pythagorean Triple ($a^2 + b^2 = c^2$). Prove that abc is even. Prove further that either exactly one of a, b and c are even, or all three are.

2. Seven coins start on a table all heads up. On any move, you may turn over any 4 of them. Is it ever possible to get all the coins tails up?

3. One hundred black checkers and 100 red checkers are placed horizontally in a row with the first and last checker being black. Prove that there is an integer n for which there are exactly as many red checkers as black checkers among the checkers numbered 1 to n.

4. A large house has a fireplace in any room with an odd number of doors. There is only one entrance to the house. Prove that one can enter the house and get to a room with a fireplace.

5. (1973 Putnam A1) Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point (not necessarily one of the given nine) on the interior of one of the line segments joining two of these points.

6. (2002 Putnam A3) Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

7. (2003 UIUC Mock Putnam). A *binary partition* of an integer is a partition of the integer into parts which are a nonnegative power of 2 (with repetition allowed). Let b_n be the number of binary partitions of n. For example, $b_5 = 4$, as 5 can be represented as $4 + 1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$, and $1 + 1 + 1 + 1 + 1$. Show that b_n is even for all $n \geq 2$.