Four Probability Conundra

1. (Mild) You are on a game show, where the host gives you the choice of three doors. Behind two of the doors are goats, behind the third is a car. You will choose a door, then the host will open one of the doors you did not choose and reveal a goat. You will then be given the chance to switch to the other unopened door. Should you switch, should you stay, or should it matter? (The host knows in advance where the car is located, and, if able to choose between two goats after you've picked your door, will do so randomly).

2. (Medium) You are on the same game show. This time, the host has no advance knowledge of the location of the car. He will choose to open a door that you did not open, and if it reveals the car, you lose automatically. If it reveals the car, you're given the same option of switching or sticking. Should you switch, should you stay, or should it matter?

3. (Hot) There are two of you on the show now. Each of you picks a different door. At least one of you has picked a goat, so the host will reveal a goat behind one of your two doors, and send that player home. (As always, if the host has an option of whom to excuse, he will do so randomly). The remaining player will be given the option of sticking or switching. Should they stick, should they switch, or should it matter?

4. (Caliente) (From Stan Wagon's Problem of the Week) Alice and Bob face three doors marked 1, 2, 3. Behind the doors are placed, randomly, a car, a key, and a goat. The couple wins the car if Bob finds the car and Alice finds the key. First Bob (with Alice removed from the scene) will open a door; if the car is not behind it he can open a second door. If he fails to find the car, they lose. If he does find the car, then all doors are closed and Alice gets to open a door in the hope of finding the key and, if not, trying again with a second door. Alice and Bob do not communicate except to make a plan beforehand. What is their best strategy?

Probability Problems

1. Alice and Bob are taking turns tossing a coin, with Alice going first. The first person to toss a head wins. What is a fair price for Alice to pay to play, if Bob is paying \$1 to play? What if, instead of being fair, the coin lands heads with probability p, with $0 < p < 1$?

2. (a) Players A and B play the following game by tossing a coin: If two heads come up in a row, A wins. If two tails come up in a row, B wins. What is the probability that A wins?

   (b) Same set up as in (a), but now A wins if three heads are tossed, and B wins if two heads are followed by a tail.

3. Suppose we toss a dart at a square dartboard. What is the probability that the dart lands closer to the center than to any corner? What is the probability that it lands closer to the center than to any edge?

4. Suppose I pass back a set of n exams to n students at random. What is the probability that any given student gets their own exam back. What is the expected number of students that get their own exam back?

5. A zombie lands in Walla Walla set on infecting people. With probability 1/3, she will infect either 0, 1, or 2 people. Each zombie in turn will infect, with equal probability, either 0, 1 or 2 people.

   (a) What is the probability that the zombies will eventually stop infecting new people? (Assume an unlimited supply of people)

   (b) On average, how many people does each zombie infect?

   (c) What is the apparent contradiction between parts (a) and (b)?

6. (a) Suppose that we have three people facing off in a 'truel' (a three-way duel). Player A has a 1/3 probability of hitting any target they fire at. Player B has a 2/3 chance. Player C always hits the target. They will take turns, and, in fairness, they decide that player A should get to shoot first. What should they do?

   (b) Same game, but now A, B, and C all have the same 1/3 chance of hitting their target.

7. We have a box of $N + 1$ bags of N candies each. In the first bag, all candies are orange. In the second, $N - 1$ candies are orange, and one is lemon. In the third bag, $N - 2$ candies are orange, and two are lemon, and so on. We take one bag at random and try the candy, which turns out to be lemon flavored. We then draw a second candy from the same bag. What is the probability that it is also lemon flavored?

8. Suppose that we choose numbers X and Y independently and at random on the interval (0,1). Compute the probability that the closest integer to X/Y is even. Express your answer as $r + s\pi$ where r and s are rational numbers.