

Compendium of Problems A1 and B1 1992-2016

1. Prove that  $f(n) = 1 - n$  is the only integer-valued function defined on the integers that satisfies the following conditions.

- (i)  $f(f(n)) = n$ , for all integers  $n$ ;
- (ii)  $f(f(n + 2) + 2) = n$  for all integers  $n$ ;
- (iii)  $f(0) = 1$ .

2. Let  $S$  be a set of  $n$  distinct real numbers. Let  $A_S$  be the set of numbers that occur as averages of two distinct elements of  $S$ . For a given  $n \geq 2$ , what is the smallest possible number of elements in  $A_S$ ?

3. The horizontal line  $y = c$  intersects the curve  $y = 2x - 3x^3$  in the first quadrant as in the figure. Find  $c$  so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the  $y$ -axis, the line  $y = c$  and the curve; the other lies under the curve and above the line  $y = c$  between their two points of intersection.]

4. Find the smallest positive integer  $n$  such that for every integer  $m$  with  $0 < m < 1993$ , there exists an integer  $k$  for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

5. Suppose that a sequence  $a_1, a_2, a_3, \dots$  satisfies  $0 < a_n \leq a_{2n} + a_{2n+1}$  for all  $n \geq 1$ . Prove that the series  $\sum_{n=1}^{\infty} a_n$  diverges.

6. Find all positive integers  $n$  that are within 250 of exactly 15 perfect squares.
7. Let  $S$  be a set of real numbers which is closed under multiplication (that is, if  $a$  and  $b$  are in  $S$ , then so is  $ab$ ). Let  $T$  and  $U$  be disjoint subsets of  $S$  whose union is  $S$ . Given that the product of any *three* (not necessarily distinct) elements of  $T$  is in  $T$  and that the product of any three elements of  $U$  is in  $U$ , show that at least one of the two subsets  $T, U$  is closed under multiplication.

8. For a partition  $\pi$  of  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ , let  $\pi(x)$  be the number of elements in the part containing  $x$ . Prove that for any two partitions  $\pi$  and  $\pi'$ , there are two distinct numbers  $x$  and  $y$  in  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  such that  $\pi(x) = \pi(y)$  and  $\pi'(x) = \pi'(y)$ . [A *partition* of a set  $S$  is a collection of disjoint subsets (parts) whose union is  $S$ .]
9. Find the least number  $A$  such that for any two squares of combined area 1, a rectangle of area  $A$  exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.
10. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of  $\{1, 2, \dots, n\}$  which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.
11. A rectangle,  $HOMF$ , has sides  $HO = 11$  and  $OM = 5$ . A triangle  $ABC$  has  $H$  as the intersection of the altitudes,  $O$  the center of the circumscribed circle,  $M$  the midpoint of  $BC$ , and  $F$  the foot of the altitude from  $A$ . What is the length of  $BC$ ?
12. Let  $\{x\}$  denote the distance between the real number  $x$  and the nearest integer. For each positive integer  $n$ , evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here  $\min(a, b)$  denotes the minimum of  $a$  and  $b$ .)

13. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
14. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for  $x > 0$ .

15. Right triangle  $ABC$  has right angle at  $C$  and  $\angle BAC = \theta$ ; the point  $D$  is chosen on  $AB$  so that  $|AC| = |AD| = 1$ ; the point  $E$  is chosen on  $BC$  so that  $\angle CDE = \theta$ . The perpendicular to  $BC$  at  $E$  meets  $AB$  at  $F$ . Evaluate  $\lim_{\theta \rightarrow 0} |EF|$ .
16. Let  $A$  be a positive real number. What are the possible values of  $\sum_{j=0}^{\infty} x_j^2$ , given that  $x_0, x_1, \dots$  are positive numbers for which  $\sum_{j=0}^{\infty} x_j = A$ ?
17. Let  $a_j, b_j, c_j$  be integers for  $1 \leq j \leq N$ . Assume for each  $j$ , at least one of  $a_j, b_j, c_j$  is odd. Show that there exist integers  $r, s, t$  such that  $ra_j + sb_j + tc_j$  is odd for at least  $4N/7$  values of  $j$ ,  $1 \leq j \leq N$ .
18. Let  $k$  be a fixed positive integer. The  $n$ -th derivative of  $\frac{1}{x^k-1}$  has the form  $\frac{P_n(x)}{(x^k-1)^{n+1}}$  where  $P_n(x)$  is a polynomial. Find  $P_n(1)$ .
19. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?
20. Let  $n$  be a fixed positive integer. How many ways are there to write  $n$  as a sum of positive integers,  $n = a_1 + a_2 + \dots + a_k$ , with  $k$  an arbitrary positive integer and  $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$ ? For example, with  $n = 4$  there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.
21. Consider a set  $S$  and a binary operation  $*$ , i.e., for each  $a, b \in S$ ,  $a * b \in S$ . Assume  $(a * b) * a = b$  for all  $a, b \in S$ . Prove that  $a * (b * a) = b$  for all  $a, b \in S$ .
22. Let  $n$  be an even positive integer. Write the numbers  $1, 2, \dots, n^2$  in the squares of an  $n \times n$  grid so that the  $k$ -th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is

one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

23. Do there exist polynomials  $a(x), b(x), c(y), d(y)$  such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

24. Basketball star Shanille O'Keal's team statistician keeps track of the number,  $S(N)$ , of successful free throws she has made in her first  $N$  attempts of the season. Early in the season,  $S(N)$  was less than 80% of  $N$ , but by the end of the season,  $S(N)$  was more than 80% of  $N$ . Was there necessarily a moment in between when  $S(N)$  was exactly 80% of  $N$ ?

25. Let  $P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0$  be a polynomial with integer coefficients. Suppose that  $r$  is a rational number such that  $P(r) = 0$ . Show that the  $n$  numbers

$$c_nr, c_nr^2 + c_{n-1}r, c_nr^3 + c_{n-1}r^2 + c_{n-2}r, \dots, c_nr^n + c_{n-1}r^{n-1} + \cdots + c_1r$$

are integers.

26. Show that every positive integer is a sum of one or more numbers of the form  $2^r3^s$ , where  $r$  and  $s$  are nonnegative integers and no summand divides another. (For example,  $23 = 9 + 8 + 6$ .)

27. Find a nonzero polynomial  $P(x, y)$  such that  $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$  for all real numbers  $a$ . (Note:  $\lfloor \nu \rfloor$  is the greatest integer less than or equal to  $\nu$ .)

28. Find the volume of the region of points  $(x, y, z)$  such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

29. Show that the curve  $x^3 + 3xy + y^3 = 1$  contains only one set of three distinct points,  $A$ ,  $B$ , and  $C$ , which are vertices of an equilateral triangle, and find its area.
30. Find all values of  $\alpha$  for which the curves  $y = \alpha x^2 + \alpha x + \frac{1}{24}$  and  $x = \alpha y^2 + \alpha y + \frac{1}{24}$  are tangent to each other.
31. Let  $f$  be a polynomial with positive integer coefficients. Prove that if  $n$  is a positive integer, then  $f(n)$  divides  $f(f(n) + 1)$  if and only if  $n = 1$ . [Editor's note: one must assume  $f$  is nonconstant.]
32. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that  $f(x, y) + f(y, z) + f(z, x) = 0$  for all real numbers  $x$ ,  $y$ , and  $z$ . Prove that there exists a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x, y) = g(x) - g(y)$  for all real numbers  $x$  and  $y$ .
33. What is the maximum number of rational points that can lie on a circle in  $\mathbb{R}^2$  whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)
34. Let  $f$  be a real-valued function on the plane such that for every square  $ABCD$  in the plane,  $f(A) + f(B) + f(C) + f(D) = 0$ . Does it follow that  $f(P) = 0$  for all points  $P$  in the plane?
35. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

36. Given a positive integer  $n$ , what is the largest  $k$  such that the numbers  $1, 2, \dots, n$  can be put into  $k$  boxes so that the sum of the numbers in each box is the same? [When  $n = 8$ , the example  $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$  shows that the largest  $k$  is at least 3.]

37. Is there an infinite sequence of real numbers  $a_1, a_2, a_3, \dots$  such that

$$a_1^m + a_2^m + a_3^m + \dots = m$$

for every positive integer  $m$ ?

38. Let  $d_1, d_2, \dots, d_{12}$  be real numbers in the open interval  $(1, 12)$ . Show that there exist distinct indices  $i, j, k$  such that  $d_i, d_j, d_k$  are the side lengths of an acute triangle.

39. Let  $S$  be a class of functions from  $[0, \infty)$  to  $[0, \infty)$  that satisfies:

- (i) The functions  $f_1(x) = e^x - 1$  and  $f_2(x) = \ln(x + 1)$  are in  $S$ ;
- (ii) If  $f(x)$  and  $g(x)$  are in  $S$ , the functions  $f(x) + g(x)$  and  $f(g(x))$  are in  $S$ ;
- (iii) If  $f(x)$  and  $g(x)$  are in  $S$  and  $f(x) \geq g(x)$  for all  $x \geq 0$ , then the function  $f(x) - g(x)$  is in  $S$ .

Prove that if  $f(x)$  and  $g(x)$  are in  $S$ , then the function  $f(x)g(x)$  is also in  $S$ .

40. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

41. For positive integers  $n$ , let the numbers  $c(n)$  be determined by the rules  $c(1) = 1$ ,  $c(2n) = c(n)$ , and  $c(2n + 1) = (-1)^n c(n)$ . Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

42. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about  $x = 0$  is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

43. A *base 10 over-expansion* of a positive integer  $N$  is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with  $d_k \neq 0$  and  $d_i \in \{0, 1, 2, \dots, 10\}$  for all  $i$ . For instance, the integer  $N = 10$  has two base 10 over-expansions:  $10 = 10 \cdot 10^0$  and the usual base 10 expansion  $10 = 1 \cdot 10^1 + 0 \cdot 10^0$ . Which positive integers have a unique base 10 over-expansion?

44. Let  $A$  and  $B$  be points on the same branch of the hyperbola  $xy = 1$ . Suppose that  $P$  is a point lying between  $A$  and  $B$  on this hyperbola, such that the area of the triangle  $APB$  is as large as possible. Show that the region bounded by the hyperbola and the chord  $AP$  has the same area as the region bounded by the hyperbola and the chord  $PB$ .
45. Let  $f$  be a three times differentiable function (defined on  $\mathbb{R}$  and real-valued) such that  $f$  has at least five distinct real zeros. Prove that  $f + 6f' + 12f'' + 8f'''$  has at least two distinct real zeros.
46. Find the smallest positive integer  $j$  such that for every polynomial  $p(x)$  with integer coefficients and for every integer  $k$ , the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j} p(x)|_{x=k}$$

(the  $j$ -th derivative of  $p(x)$  at  $k$ ) is divisible by 2016.

47. Let  $x_0, x_1, x_2, \dots$  be the sequence such that  $x_0 = 1$  and for  $n \geq 0$ ,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function  $\ln$  is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.