Four Probability Conundra

1. (Mild) You are on a game show, where the host gives you the choice of three doors. Behind two of the doors are goats, behind the third is a car. You will choose a door, then the host will open one of the doors you did not choose and reveal a goat. You will then be given the chance to switch to the other unopened door. Should you switch, should you stay, or should it matter? (The host knows in advance where the car is located, and, if able to choose between two goats after you've picked your door, will do so randomly).

2. (Medium) You are on the same game show. This time, the host has no advance knowledge of the location of the car. He will choose to open a door that you did not open, and if it reveals the car, you lose automatically. If it reveals the car, you're given the same option of switching or sticking. Should you switch, should you stay, or should it matter?

3. (Hot) There are two of you on the show now. Each of you picks a different door. At least one of you has picked a goat, so the host will reveal a goat behind one of your two doors, and send that player home. (As always, if the host has an option of whom to excuse, he will do so randomly). The remaining player will be given the option of sticking or switching. Should they stick, should they switch, or should it matter?

4. (SuperDuperSpicy) (From Stan Wagon's Problem of the Week) Alice and Bob face three doors marked 1, 2, 3. Behind the doors are placed, randomly, a car, a key, and a goat. The couple wins the car if Bob finds the car and Alice finds the key. First Bob (with Alice removed from the scene) will open a door; if the car is not behind it he can open a second door. If he fails to find the car, they lose. If he does find the car, then all doors are closed and Alice gets to open a door in the hope of finding the key and, if not, trying again with a second door. Alice and Bob do not communicate except to make a plan beforehand. What is their best strategy?

Probability Problems

1. Alice and Bob are taking turns tossing a coin, with Alice going first. The first person to toss a head wins. What is a fair price for Alice to pay to play, if Bob is paying $1 to play?

2. Suppose we toss a dart at a square dartboard. What is the probability that the dart lands closer to the center than to any corner?

3. A prisoner is given twelve black balls, twelve white balls, and two urns. She may distribute the balls however she likes, but must place at least one ball in each urn. The guard will enter the room, choose an urn at random, and choose a ball from that urn. The prisoner is freed if the ball is white. How should the prisoner distribute the balls so as to maximize her chances of freedom?

4. In the border of a perfectly circular piece of wood, we choose 3 points at random to place legs and make a table. What is the probability that the table will stand without falling? What if we replace 3 by n?

5. (2004 Putnam A1) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

6. (R. Stanley) You have $n > 1$ numbers $0, 1, \ldots, n$ arranged in a circle. A random walker starts at 0 and at each step moves at random to one of its two nearest neighbors. For each i, compute the probability p_i that when the walker is at point i for the first time, all other points have been previously visited, that is, that i is the last new point. (eg, $p_0 = 0$)

7. (1993 Putnam A3) Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.