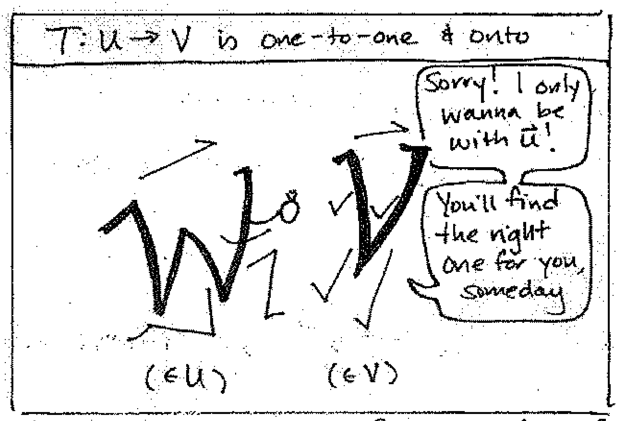
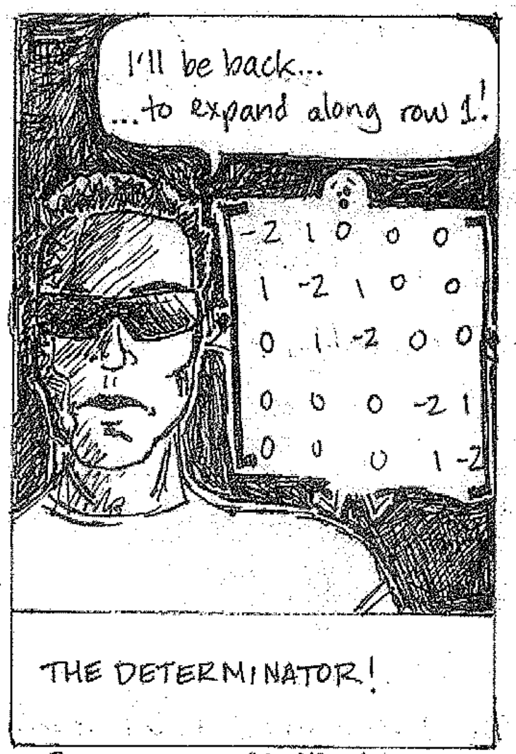
KEY

Math 300: Midterm Exam
Fall 2008

This exam is closed book and closed notes. You may use a calculator only for basic arithmetic. You have until 5 minutes before the hour to finish the in-class portion.



2006 © COURTNEY GIBBONS



2007 ©COURTNEY GIBBONS

21

1. True or False. Give a brief justification in each case.

(a) If A is an $m \times n$ matrix with $m \leq n$, then $A\mathrm{x} = \mathbf{0}$ has infinitely many solutions.

False! $m \left[\quad \right]_{n}$ $m \geq n \Rightarrow$ free variables only if $m \geq n$.
$m = n$ might not have free variables.
so false

(b) If S is a linearly dependent set of vectors, and S' is a subset of S, then S' is also a linearly dependent set of vectors.

False: $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ dep. set
$\{\vec{v}_1\}$ if $\vec{v}_1 \neq 0$ is an indep set.

(c) If A and B are $n \times n$ matrices and $AB = I_n$, then $BA = I_n$.

True! In this case $A = B^{-1}$, so inverse matrices commute.

(d) If $T(\mathrm{x}) = A\mathrm{x}$ is both one-to-one and onto, then A must be a square matrix.

True! A must have a pivot in every row, every column, so rows = columns.

(e) If there exists a b such that $A\mathrm{x} = \mathrm{b}$ has only one solution, then $A\mathrm{x} = \mathbf{0}$ has only the trivial solution.

True! In this case $T(\vec{x}) = A\vec{x}$ is 1-1, so A can have no free variables.

(f) If A and B are row equivalent square matrices, then $\det(A) = \det(B)$

False! Row reduction affects the determinant

(g) If A and B are square matrices, then $\det(AB) = \det(BA)$.

True! $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$

12

2. Each of the following sentences are mathematically incorrect. They are not false in their intended meaning, rather, they abuse language in some way or another. Correct each sentence so that each gets its meaning across and makes mathematical sense.

   (a) If $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$, then ~~$A$ spans $\mathbb{R}^m$~~.

   , then the columns of A span $\mathbb{R}^m$

   (b) Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ ~~is a matrix.~~

   has an associated $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$.

   (c) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if it ~~has a unique solution~~.

   $T(\vec{x}) = \vec{b}$ has a unique solution whenever $T(\vec{x}) = \vec{b}$ is consistent

   (d) A consistent system of equations has a unique solution if and only if ~~it~~ has linearly independent columns.

   The associated matrix

27

3. Suppose that T is a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$. Suppose further that $T(1,0,0) = (1,2)$, $T(1,0,1) = (8,14)$ and that $T(1,2,1) = (12,20)$.

(a) Find the standard matrix for T and use it to compute $T(1,2,3)$.

$T(1,0,0) = (1,2)$

$T(1,0,1) = (8,14) \Rightarrow T(0,0,1) = (8,14) - (1,2) = (7,12)$

$\cancel{T(1,2,1)}\ T(1,2,1) = (12,20) \Rightarrow T(0,2,0) = (12,20) - (8,14) = (4,6)$

$\Rightarrow T(0,1,0) = (2,3)$

$$\begin{array}{cccc} & T(1,0,0) & T(0,1,0) & T(0,0,1) \\ A = & \left[\begin{array}{c} 1 \\ 2 \end{array}\right. & \begin{array}{c} 2 \\ 3 \end{array} & \left.\begin{array}{c} 7 \\ 12 \end{array}\right] \end{array}$$

$$T(1,2,3) = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 3 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+4+21 \\ 2+6+36 \end{bmatrix} = \begin{bmatrix} 26 \\ 44 \end{bmatrix}$$

(b) Is T a one-to-one map? If not, find at least two solutions to $T(\mathbf{x}) = \mathbf{0}$ (You may find all solutions if that is easier for you).

T cannot be 1-1 as it does not have a pivot in every column.

$$\left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 2 & 3 & 12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

x_3 free

$x_1 + 3x_3 = 0$

$x_2 + 2x_3 = 0$

all solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

24

4. (a) Find an LU factorization of

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -6 & 6 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -12 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -12 \end{bmatrix}$$

(b) Use your decomposition to easily calculate $\det(A)$.

$$\det(A) = \det(LU) = \det(L)\det(U) =$$
$$1 \cdot (3)(-3)(-12) = \underline{108}$$

(10)

5. Calculate $\det(A)$, where

cofactor here

$$A = \begin{bmatrix} 3 & -1 & 0 & 2 & 1 \\ 4 & 0 & 0 & 1 & 2 \\ 0 & 2 & -3 & 4 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 4 & 2 & 0 & 3 & 1 \end{bmatrix}$$

$$\det(A) = -3 \begin{vmatrix} 3 & -1 & 2 & 1 \\ 4 & 0 & 1 & 2 \\ 0 & -2 & 0 & 0 \\ 4 & 2 & 3 & 1 \end{vmatrix} \leftarrow \quad \rightarrow (-3)(-(-2)) \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= (-3)(-(-2))\left[3(1-6) - 2(4-8) + 1(12-4)\right]$$

$$= (-3)(2)\left[-15+8+8\right] = \underline{-6}$$

6. List as many equivalent statements (at least 5, at most 10) to the following statement as you can:

(10) +

A is an invertible matrix

a) A has lin indep columns
b) A has lin indep rows
c) $T(\vec{x}) = A\vec{x}$ is 1-1
d) $T(\vec{x}) = A\vec{x}$ is onto
e) $T(\vec{x}) = A\vec{x}$ is invertible
f) A has a pivot in every row
g) $\mathrm{rref}(A) = I_n$
h) $\det(A) \neq 0$
i) $\exists$ $n \times n$ C s.t. $CA = I_n$
j) $\exists$ $n \times n$ D s.t. $AD = I_n$

etc.