Math 300: Quiz the ~~First~~ Second Key

9/25/08

This exam is closed book and closed notes. You may not use a calculator. You have until 5 minutes before the hour to finish the in-class portion. The take-home portion is due Tuesday at the beginning of class.

1. True or False. Give a brief justification in each case.

5

(a) If $\mathbf{y}$ and $\mathbf{z}$ are solutions to $A\mathbf{x} = \mathbf{0}$, then so is $\mathbf{y} + \mathbf{z}$.

True: $A(\vec{y}+\vec{z}) = A\vec{y} + A\vec{z} = \vec{0} + \vec{0} = \vec{0}$

(b) If there is a set of 3 vectors with 4 entries in each, then the set must be linearly independent.

False: $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -3 \\ -7 \end{bmatrix}$ is lin. dependent

(c) For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{0}$ is in the range of T.

True $T(\vec{0}) = \vec{0}$

(d) If A has a pivot in every row, then $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

False. pivot in every row $\to$ onto

" " " column $\to$ one-to-one

(e) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ must be onto

False $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 0 & -6 \end{bmatrix}$ doesn't have a pivot in every ~~col~~ row.

2. Consider the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 1 & -3 & 0 \\ -1 & -2 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$$

The row reduced echelon form of the associated *augmented* matrix is

$$\begin{array}{cccccc} P & F & P & F & P & \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{aligned} x_1 &= -2x_2 + x_4 + 1 \\ x_3 & \; -x_4 + 2 \\ x_5 &= -1 \end{aligned}$$

Use this to write the solution set of the equation in parametric vector form.

4

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_4 + 1 \\ x_2 \\ -x_4 + 2 \\ x_4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

3. What is the solution set of the homogeneous equation

3

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 1 & -3 & 0 \\ -1 & -2 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ?$$

from above, dropping the particular soln.

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

4. Show that the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 2 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 15 \\ -2 \\ 9 \\ -19 \end{bmatrix}$ are linearly dependent by row reducing an appropriate matrix, then writing $\mathbf{0}$ as a linear combination of the three vectors.

4

$$\begin{bmatrix} 1 & 5 & 15 \\ 2 & 2 & -2 \\ -1 & 1 & 9 \\ 3 & -1 & -19 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 15 \\ 0 & -8 & -32 \\ 0 & 6 & 24 \\ 0 & -16 & -64 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 15 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So by letting $x_1 = 5x_3$

$x_2 = -4x_3$

x_3 free, so $x_3 = 1$, $x_2 = -4$, $x_1 = 5$

we get

$$5\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 5 \\ 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 15 \\ -2 \\ 9 \\ -19 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Give as many equivalent conditions (at least 2 each, bonus for more) to each of the following statements as you can. BE PRECISE in your language.

(a) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto.

- For each $\vec{b}$ in $\mathbb{R}^m$, there is an $\vec{x}$ in $\mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$
- For the associated matrix A
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in every row
- The range of T is the codomain of T.

4

(b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.

- For each $\vec{b}$ in $\mathbb{R}^m$, there is at most one $\vec{x}$ in $\mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$
- $T(\vec{0}) = \vec{0}$ has only the trivial solution
- For the associated matrix A
  - The columns of A are independent
  - Each column of A is a pivot column.
- If $T(\vec{x}) = T(\vec{y})$ then $\vec{x} = \vec{y}$.