KEY

Math 300: Quiz the Third

This exam is closed book and closed notes. You may use a calculator only for basic arithmetic. You have until 5 minutes before the hour to finish the in-class portion.

1. True or False. Give a brief justification in each case.

   (a) For an $n \times m$ matrix A, the product AA^T is always defined.

   True. $(n\times m)(m\times n)$ is defined

   (b) The product of two elementary matrices is an elementary matrix

   False $\begin{bmatrix} a_{1,1} & 0 \\ 0 & b_{1,1} \end{bmatrix}$ is not elementary

   (c) If $AB = BA$ for some pair of matrices A and B, then $(AB)^{-1} = A^{-1}B^{-1}$.

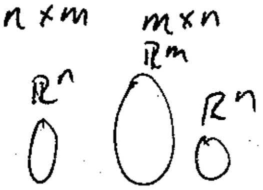
   True $(AB)^{-1} = (BA)^{-1} = A^{-1}B^{-1}$

   (d) If A and B are invertible $n \times n$ matrices, then $AB = BA$

   False. Matrices don't commute in general.

   (e) If A is an $m \times n$ matrix, and there is an $n \times m$ matrix C such that $CA = I_n$, then $n \leq m$.

   $\underset{n\times m}{C}\ \underset{m\times n}{A} = I_n \Rightarrow m \geq n$, so, true.



2. Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 9 \end{bmatrix}$$

and use it to solve $A\mathrm{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & +3 & 3 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 5 & -1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -20 & 3 & 5 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 5 & -1 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -20 & 3 & 5 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{bmatrix}$$

$$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -20 & 3 & 5 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -2 \end{bmatrix}$$

3. Suppose that A is an $n \times n$ matrix. Some of the following are equivalent to the statement "A is an invertible matrix". Which are, which are not, and why? (Pay attention here: THIS IS NOT A TRUE-FALSE QUESTION.)

(a) $T(\mathrm{x}) = A\mathrm{x}$ is one-to-one.

This is equiv. to A is invertible by the IMT.

(b) A has a column of zeros in the reduced echelon form.

A is not invertible in this case (no lin indep. columns)

(c) $A\mathrm{x} = \mathbf{0}$ has the trivial solution.

This is always true and is not enough to say that A is invertible.

(d) The columns of A span $\mathbb{R}^n$.

~~This is true~~. $T(\vec{x}) = A\vec{x}$ is onto & is part of the IMT.
A is invertible

(e) A is a product of elementary matrices.

A is invertible. We showed this in class.

(f) $A(\lambda \mathrm{x}) = \lambda A\mathrm{x}$.

Again, this is true for all A, invertible or not so we cannot conclude that A is invertible

4. In the homework, you computed the $n \times n$ matrices of the form $\mathbf{u}\mathbf{v}^{\mathrm{T}}$, where $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ column vectors. How many pivots does this product matrix have? Explain why. (Feel free to compute with a couple of specific examples to get started here. That is, make up a $\mathbf{u}$, and a $\mathbf{v}$, and see what happens).

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & \cdots & u_n v_n \end{bmatrix}$$

Columns are all scalar multiples of eachother, therefore uv^T has only 1 pivot.

$$\text{rref}(uv^T) = \begin{bmatrix} 1 & v_2/v_1 & v_3/v_1 & \cdots & v_n/v_1 \\ & & 0 & & \end{bmatrix}$$