KEY

Math 300: Quiz the Fourth

This exam is closed book and closed notes. You may not use a calculator. You have until 5 minutes before the hour to finish the in-class portion. The take-home portion is due Monday at the beginning of class.

1. True or False. Give a brief justification in each case.

   (a) Any plane in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

   False - plane must go through the origin

   (b) The set of all continuous real-valued functions is a vector space.

   True → one of our examples

   (c) A vector space can have more than one zero vector.

   False → zero vector is unique

   (d) A linear transformation is one-to-one if its kernel is empty.

   ~~False~~ → Kernel always has $\vec{0}$. → so the premise is always false

   True

   (e) Every vector space with more than one vector has more than one subspace.

   True. $\{\vec{0}\}$ and V are subspaces of V.

2. Use Cramer's Rule to solve $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

and $b = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$

$$\det A = 2(-6-1) - 1(4-3) + 5(2-(-9))$$

$$= -14 - 1 + 55 = 40$$

$$A_1(\vec{b}) = \begin{bmatrix} 0 & 1 & 5 \\ 0 & -3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \to \det = 5(1+15) = 80 \qquad x_1 = \frac{80}{40} = 2$$

$$A_2(\vec{b}) = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 0 & 1 \\ 3 & 5 & 2 \end{bmatrix} \to \det = -5(2-10) = +40 \qquad x_2 = \frac{40}{40} = 1$$

$$A_3(\vec{b}) = \begin{bmatrix} 2 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & 1 & 5 \end{bmatrix} \to \det = 5(-6-2) = -40 \qquad x_3 = \frac{-40}{40} = -1$$

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

3. For each given vector space V and subset H of V, determine whether H is a subspace of V.

(a) $V = \mathbb{R}^3$, $H = \left\{ \begin{bmatrix} a \\ 2b \\ 3a - b \end{bmatrix} ; a, b \in \mathbb{R} \right\}$

yes, H is the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$

So closed under +, sc. mult, has $\vec{0}$ vector

(b) $V = \mathbb{R}^4$, $H = \left\{ \begin{bmatrix} a \\ b \\ c \\ b - a - 1 \end{bmatrix} ; a, b, c \in \mathbb{R} \right\}$

No. $\begin{bmatrix} a \\ b \\ c \\ b-a-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a=0 \\ b=0, \end{matrix}$ but then $b-a-1 = -1$,

So $\vec{0} \notin H$.

(c) $V = \mathbb{P}_3$, H is the set of polynomials whose coefficients sum to 0.

$\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + a_1 + a_2 + a_3 = 0 \}$

Yes

$$\begin{array}{l} a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ + \; b_0 + b_1 x + b_2 x^2 + b_3 x^3 \\ \hline (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3 \end{array}$$

$a_0 + b_0 + a_1 + b_1 + a_2 + b_2 + a_3 + b_3 =$

$\underbrace{(a_0 + a_1 + a_2 + a_3)}_{0} + \underbrace{(b_0 + b_1 + b_2 + b_3)}_{0}$ ✓

$\lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \lambda a_3 x^3 \rightarrow \lambda(a_0 + a_1 + a_2 + a_3) = \lambda(0) = 0$

$0 \rightarrow$ coeff's sum to 0.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & 5 & 1 & 4 \\ 2 & 6 & 0 & 2 & 1 & 3 \\ -1 & -3 & 2 & 3 & 0 & 1 \\ 3 & 9 & 5 & 13 & 1 & 9 \end{bmatrix}$$

and its row reduced echelon form

$$R = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Determine if $\begin{bmatrix} -5 \\ 1 \\ -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \vec{x}$ is in nul(A).

Multiplying $A\vec{x}$ gives $\vec{0}$, so yes $\vec{x} \in \text{nul}(A)$
(or $R\vec{x}$ gives $\vec{0}$)

(b) Give two nonzero vectors that are in col(A) that are not themselves columns of A.

$c_1 + c_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 8 \end{bmatrix}$ – both in col(A), as is any lin. comb. of vectors.

$c_3 + c_5 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 6 \end{bmatrix}$ –

(Note: Be explicit about your methods on this one!!)

Math 300: Quiz the Fourth-Take Home Key

1. An $n \times n$ matrix A is symmetric if $A = A^T$. Prove that the set of symmetric $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrices, and find a spanning set for this subspace.

   SOLUTION: Let A and B be symmetric matrices. Then $(A+B)^T = A^T + B^T = A + B$, so symmetric matrices are closed under addition. $(\lambda A)^T = \lambda(A^T) = \lambda A$, so symmetric matrices are closed under scalar multiplication. The all zeroes matrix is symmetric, so the symmetric matrices form a subspace of the space of matrices.

   To form a spanning set for this set, consider the set of matrices $M(ij)$, which is an $n \times n$ matrix with a 1 in the (i, j) position and 0's everywhere else. The subspace of symmetric matrices is spanned by all matrices of the form $(M(ij)+M(ji))$ (that is, matching 1's opposite the diagonal and 0's everywhere else), together with those of the form $M(ii)$ (those with a 1 in a diagonal position).

2. Suppose that A is an $n \times n$ matrix. Prove (carefully) that $A^2 = 0$ if and only if the column space of A is a subspace of the null space of A.

   SOLUTION: Suppose first that $A^2 = 0$, and let $\mathbf{y}$ be in the column space of A. Then $\mathbf{y} = A\mathbf{x}$ for some x. But then $A\mathbf{y} = A^2\mathbf{x} = 0\mathbf{x} = 0$, so $\mathbf{y}$ is also in the null space of A. That the column space is a subspace of the null space follows immediately from the fact that it is a subspace of $\mathbb{R}^n$ (and so is closed under addition, etc.), we only needed to prove that it was a *subset* of the null space.

   Suppose instead that the column space is a subspace of the null space. Consider $A^2\mathbf{x}$ for any vector $\mathbf{x} \in \mathbb{R}^n$. $A^2\mathbf{x} = A(A\mathbf{x})$. Let $\mathbf{y} = A\mathbf{x}$. Then $\mathbf{y}$ is in the column space of $\mathbf{x}$ (by definition), thus it is also in the null space of A (by assumption). Thus $A\mathbf{y} = 0$, or $A(A\mathbf{x}) = A^2\mathbf{x} = 0$ for all x in $\mathbb{R}^n$, so A^2 must be the 0 matrix.

3. Consider the linear transformation $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by $T(p(x)) = p(x) + p'(x)$. Prove that T is one-to-one and onto, and find an inverse transformation for T.

   SOLUTION: We calculate for an arbitrary $p(x) = a_0+a_1x+a_2x^2+a_3x^3$, $T(p(x) = (a_0 + a_1x + a_2x^2 + a_3x^3) + (a_1 + 2a_2x + 3a_3x^2) = (a_0 + a_1) + (a_1 + 2a_2)x + (a_2 + 3a_3)x^2 + a_3x^3$.

   We first show that T is 1-1. Assume that $T(p(x)) = 0$. Setting coefficients equal gives the following homogeneous system.

$$a_0 + a_1 = 0$$
$$a_1 + 2a_2 = 0$$
$$a_2 + 3a_3 = 0$$
$$a_3 = 0$$

   which has only the trivial solution. Thus, $T(p(x)) = 0$ implies that $p(x) = 0$, thus T is one-to-one.

   To show that T is onto, and in the process, create an inverse transformation for T, we show that the equation $T(p(x)) = c_0 + c_1x + c_2x^2 + c_3x^3$ has a solution for all choices of c_0, c_1, c_2, c_3. But this is the same as solving the system of equations

$$a_0 + a_1 = c_0$$
$$a_1 + 2a_2 = c_1$$
$$a_2 + 3a_3 = c_2$$
$$a_3 = c_3$$

which has the unique solution

$$(a_0, a_1, a_2, a_3) = (c_0 - c_1 + 2c_2 - 6c_3, c_1 - 2c_2 + 6c_3, c_2 - 3c_3, c_3)$$

So we have shown the map to be onto and have given a formula for the inverse.

$$T^{-1}(c_0+c_1x+c_2x^2+c_3x^3) = (c_0-c_1+2c_2-6c_3)+(c_1-2c_2+6c_3)x+(c_2-3c_3)x^2+c_3x^3$$