KEY

Math 300: Quiz the Last

This exam is closed book and closed notes. You may use a calculator for arithmetic only. You have until 5 minutes before the hour to finish the in-class portion. The take-home portion is due Tuesday at the beginning of class.

1. True or False. Give a brief justification in each case.

   (Assume that V is an n-dimensional vector space with basis β and W is an m-dimensional vector space with basis γ) .

   (a) If $T : V \to W$ is a linear transformation, then the matrix for T relative to β and γ is an $n \times m$ matrix.

   $T: \underset{n}{V} \to \underset{m}{W}$ $\quad$ $[T]$ is $m \times n$, so

   FALSE

   (b) If $T : V \to W$ has as its matrix relative to β and γ an invertible matrix, then $m = n$.

   True. $\quad$ T must be 1-1, onto

   (c) If $3 + i$ is an eigenvalue of a matrix A, then so is $-3 + i$

   False. $\quad$ $3 - i$ is an eigenvalue in this case

   (d) If $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$.

   True $\quad$ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$, only $\vec{0}$ has magnitude 0.

   (e) If $\mathbf{y}$ is a vector in $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$, then $\mathbf{y} \perp \mathbf{proj}_W(\mathbf{y})$

   False $\quad$ $(\vec{y} - \text{proj}_W \vec{y}) \perp \text{proj}_W \vec{y}$

2. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_3$ given by

$$T(p(x)) = \int_0^x p(t)\, dt + p(x)$$

(a) Compute $T(3x^2 + 2x + 4)$

$$T(3x^2+2x+4) = \int_0^x 3t^2+2t+4\, dt + 3x^2+2x+4$$
$$= x^3 + x^2 + 4x + 3x^2+2x+4 =$$
$$\underline{x^3+4x^2+6x+4}$$

(b) Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$. Compute the matrix of T relative to β and γ.

$$T(1) = x+1$$
$$T(x) = \frac{x^2}{2} + x$$
$$T(x^2) = \frac{x^3}{3} + x^2$$

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(c) Use this matrix to verify your calculation in part (a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 1 \end{bmatrix} = x^3+4x^2+6x+4$$

✓ ☺ etc.

3. Consider the matrix

$$A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$$

(a) Use the characteristic polynomial to prove that $\lambda = 2 - 3i$ is an eigenvalue of A.

char poly $\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 3-\lambda & -5 \\ 2 & 1-\lambda \end{bmatrix}$

$= (3-\lambda)(1-\lambda) + 10$

$= 3 - 4\lambda + \lambda^2 + 10$

$= \lambda^2 - 4\lambda + 13$

roots:

$\frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 13}}{2}$

$= \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$

$= \frac{4 \pm 6i}{2} = 2 \pm 3i$ ✓

(b) The corresponding eigenvector to λ is $\begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$. Use this to write $A = PCP^{-1}$, where C is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$P = \begin{bmatrix} \mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v} \\ \cancel{\mathrm{Im}\,\vec{v}} & \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$

$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $\lambda = a - bi$, so $a = 2$, $\lambda = 3$

$A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$

4. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. Express $\mathbf{y} = \mathbf{x} + \mathbf{z}$, where $\mathbf{x}$ is a multiple of $\mathbf{v}$ and $\mathbf{z}$ is orthogonal to $\mathbf{v}$.

$$\vec{x} = \text{proj}_{\vec{v}}\, \vec{y} = \frac{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \frac{2-2+6}{1+4+4} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 4/3 \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 7/3 \\ 5/3 \end{bmatrix}$$

Check $\begin{bmatrix} 4/3 \\ 7/3 \\ 5/3 \end{bmatrix} \circ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \left(\frac{4}{3} - \frac{14}{3} + \frac{10}{3}\right) = 0$.

5. Prove that if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

$$d(\vec{u}, \vec{v}) = d(\vec{u}, -\vec{v})$$

$$\|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\|$$

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2$$

$$(\vec{u} - \vec{v}) \cdot (u - \vec{v}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \cancel{\vec{u} \cdot \vec{u}} - 2\vec{u} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{v}} = \cancel{\vec{u} \cdot \vec{u}} + 2\vec{u} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{v}}$$

$$-2\vec{u} \cdot \vec{v} = 2\vec{u} \cdot \vec{v}$$

$$4\vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}.$$