KEY

Math 300: Midterm Exam
Spring 2010

This exam is closed book and closed notes. You may use a calculator only for basic arithmetic. You have until 2PM to finish the in-class portion.

1. True or False. Give a brief justification in each case.

   (a) A homogeneous system with more unknowns than equations has infinitely many solutions.

   True. More unknowns than eq'ns ⇒ atleast 1 free variable.

   (b) If a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $T(\mathbf{u}) = T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, then $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^m$.

   True $T(\vec{u}) = T(\vec{0}) = \vec{0}$ for all $\vec{u} \in \mathbb{R}^m$

   (c) If S is a linearly independent set of vectors in $\mathbb{R}^n$, then S cannot span $\mathbb{R}^n$.

   False. $\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix} \cdots \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}$ is a lin indep span set.

   (d) If an $m \times n$ matrix A has m pivots, and B is a $n \times p$ matrix, then AB has m pivots as well.

   $\mathbb{R}^p \to \mathbb{R}^n \to \mathbb{R}^m$

   ~~True $T(\vec{x}) = A\vec{x}$ is onto, as will~~

   False. eg. B could be the 0 matrix, removing all pivots.

   (e) if A and B are matrices such that AB is defined, and the first two rows of B are the same, then so are the first two rows of AB.

   False. True for columns, but not for rows.

   (f) If A is invertible, then $\det(A^2) > 0$.

   True $\det(A) \neq 0$, $\det(A^2) = \det(A)^2 > 0$

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2. Each of the following sentences are mathematically incorrect. They are not false in their intended meaning, rather, they abuse language in some way or another. Correct each sentence so that each gets its meaning across and makes mathematical sense.

   (a) If an $m \times n$ matrix A is one-to-one, then $m \geq n$.

   If $T(\vec{x}) = A\vec{x} : \mathbb{R}^n \to \mathbb{R}^m$ is 1-1, then $m \geq n$

   (b) An $n \times n$ matrix is invertible if and only if ~~it~~ span~~s~~ $\mathbb{R}^n$.

   its columns

   (c) If the columns of A are linearly independent, then ~~it~~ has only the trivial solution.

   $A\vec{x} = \vec{0}$

   (d) If a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then ~~its~~ the determinant is non-zero.

   of the associated matrix A

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3. Let

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 2 & 6 & -5 & 0 & -1 \\ -1 & -3 & 0 & 0 & 3 \end{bmatrix}$$

(a) Find the row reduced echelon form of A.

$$\begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 2 & 6 & -5 & 0 & -1 \\ -1 & -3 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 0 & 0 & -13 & -4 & -2 \\ 0 & 0 & 4 & 2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & -13 & -4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & \frac{5}{2} & 11 \end{bmatrix}$$

$-4+\frac{13}{2}=2.5$

$$\to \begin{bmatrix} 1 & 3 & 4 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 1 & 22/5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(b) Why do columns 1, 3, and 4 of A form a linearly independent set?

Columns 1, 3, 4 are pivot columns of A, thus are linearly independent as a set of vectors.

(c) Write the fifth column of A as a combination of columns 1, 3, and 4. of A

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

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4. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(a) Calculate the first few powers of A and comment on the pattern of A^n

$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$

A^3? $\quad A^4 \quad A^5$

its enough to go here

A^n contains Fibonacci numbers!

$\begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 21 & 13 \\ 13 & 8 \end{bmatrix}$

(b) Give an exact formula for $\det(A^n)$

$\det(A) = -1$

$\det(A^n) = (\det A)^n = (-1)^n$

(c) Let $B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Give a formula for B^n. (You should use block multiplication to help with some of the work here, and your answer can be in terms of A).

$B = \begin{bmatrix} A & I \\ 0 & A \end{bmatrix} \quad B^2 = \begin{bmatrix} A^2 & 2A \\ 0 & A^2 \end{bmatrix} \cdot \begin{bmatrix} A & I \\ 0 & A \end{bmatrix} = \begin{bmatrix} A^3 & 3A^2 \\ 0 & A^3 \end{bmatrix} \cdot \begin{bmatrix} A & I \\ 0 & A \end{bmatrix}$ (B^3)

$\begin{bmatrix} A & I \\ 0 & A \end{bmatrix} \begin{bmatrix} A & I \\ 0 & A \end{bmatrix} \qquad = \begin{bmatrix} A^4 & 4A^3 \\ 0 & A^4 \end{bmatrix}$

$B^n = \begin{bmatrix} A^n & nA^{n-1} \\ 0 & A^n \end{bmatrix}$

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5. Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 6 & 3 & 13 \\ -3 & 2 & 20 \end{bmatrix}$.

(a) Find an LU decomposition for A.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 6 & 3 & 13 \\ -3 & 2 & 20 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 3 & 24 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

(b) Use your decomposition to solve $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$

$$A\vec{x} = L(Ux) = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$$

$$L\vec{y} = \begin{bmatrix} 1 \\ 0 \\ -7 \end{bmatrix}$$

Find $\vec{y}$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & -7 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 9 & 0 \end{array}\right]$$

$$\begin{bmatrix} y_1 = 1 \\ y_2 = -2 \\ y_3 = 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$x_3 = 0$ $\quad x_2 = -2$

$x_1 = 1$

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$ interesting!

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6. Let A be an $n \times n$ matrix. Which of the following are *equivalent* to the statement

A is an invertible matrix.

Beware: This is not a true-false section

(a) A has a pivot in every row.

Equiv. ~~False~~. Such a matrix has lin indep col's, etc.

(b) $A\mathbf{x} = 0$ has the trivial solution.

Not ~~False~~.
Equiv ∵ $A\vec{x} = 0$ Always has the triv. sol'n

(c) A is row equivalent to an upper triangular matrix.

Not equiv ; Such a form does not guarantee invertibility (upper triangular could still have 0's on the diagonal)

(d) $T(\mathbf{x}) = A\mathbf{x}$ is a one-to-one map.

Equiv.. For maps $\mathbb{R}^n \to \mathbb{R}^n$ 1-1 ⇔ onto ⟷ invertible.

(e) For all $n \times n$ matrices B, $B \neq 0$, $AB \neq 0$ (that is, the matrix of all zeroes).

Equiv. ~~False~~. If A were not invertible, there would be a dependence relation among the columns of A $\lambda_1 \vec{c}_1 + \lambda_2 \vec{c}_2 + \ldots + \lambda_n \vec{c}_n = 0$ where not all $\lambda_i = 0$

Then $A \begin{bmatrix} \lambda_1 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_2 & \lambda_2 & & \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_n & \lambda_n & & \lambda_n \end{bmatrix} = 0$, so there would exist such a matrix.

If A were inv. $AB = 0 \Rightarrow B = 0$ ($0A^{-1}$).

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