KEY

Math 300: Quiz the Third

This exam is closed book and closed notes. You may use a calculator only for basic arithmetic. You have until 5 minutes before the hour to finish the in-class portion..

1. True or False. Give a brief justification in each case.

   (a) For an $n \times m$ matrix A, the sum $A + A^T$ is always defined.

   False : not defined if $m \neq n$

   (b) If A is an invertible matrix and B is obtained from A through elementary row operations, then B is an invertible matrix.

   True. Both row reduce to the identity matrix.

   (c) If A and B are invertible $n \times n$ matrices, then so is $A + B$

   False. Ie. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   (d) For square matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$.

   False $(A+B)(A+B) = A^2 + AB + BA + B^2$

   $AB \neq BA$ in general

   (e) If columns 1 and 3 of a matrix B are the same, then so are columns 1 and 3 of AB whenever AB is defined.

   True $\mathrm{Col}_1(AB) = A \cdot \vec{b}_1 = A \cdot \vec{b}_3 = \mathrm{col}_3(AB)$

2. Set up a system of equations to balance

$$KOH + Cl_2 = KClO_3 + KCl + H_2O$$

$$x_1\ KOH + x_2\ Cl_2 = x_3\ KClO_3 + x_4\ KCl + x_5\ H_2O$$

$K \to \quad x_1 = x_3 + x_4$

$O \to \quad x_1 = 3x_3 + x_5$

$H \to \quad x_1 = 2x_5$

$Cl \to \quad 2x_2 = x_3 + x_4$

$x_1 - x_3 - x_4 = 0$

$x_1 - 3x_3 - x_5 = 0$

$x_1 \quad - 2x_5 = 0$

$2x_2 - x_3 - x_4 = 0$

3. Translate your system to a matrix equation and explain why you must have a solution that can balance the equation. (You NEED NOT SOLVE your system).

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & -3 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 2 & -1 & -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A has at least 1 free variable $\Rightarrow$ nontrivial solution to the homogeneous eq'n.

4. Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

and use it to solve $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

Row Reduce

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & \cancel{1} & \cancel{0} \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \leftarrow 1 \quad 1 \quad -1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Check

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

✓ !! etc.

$$\vec{x} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

5. Let A be a 2×2 matrix with all positive entries. Prove that A^{-1} has two positive and two negative entries.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$a, b, c, d > 0$

$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

2 cases: $ad-bc > 0 \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$ad-bc < 0 \quad \begin{bmatrix} - & + \\ + & - \end{bmatrix}$ ✓

6. Does the above statement hold if we replace "positive" by "non-negative"?

not necessarily.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$