KEY

Math 300: Quiz the Fifth
Spring 2010

This exam is closed book and closed notes. You may use a calculator for arithmetic only. You have until 5 minutes before the hour to finish the in-class portion. The take-home portion is due Tuesday at the beginning of class.

1. True or False. Give a brief justification in each case.

   (a) A basis of a subspace H of a vector space V is any set of vectors which spans H.

   False - must be lin. indep.

   (b) A smallest linearly independent set of a finite-dimensional vector space must be a basis for that vector space.

   False largest lin. indep. or smallest spanning

   (c) The rank of a matrix plus the dimension of the column space equals the number of columns of that matrix.

   False → rank + dim nullspace.

   (d) The dimension of the vector space of $n \times n$ matrices is $2n$.

   False → $\dim = n^2$

   (e) If $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ is an eigenvector of A^2.

   True $A^2\vec{v} = A(A\vec{v}) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$.

2. (a) Use coordinate vectors to tell if the set $\{x, x^3+1, x^3-x\}$ is linearly independent in $\mathbb{P}_3$. (That is, row reduce an appropriate matrix)

Basis for $\mathbb{P}_3$ $\{1, x, x^2, x^3\}$

$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} x & & \end{array} \end{array} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3 pivots, so, yes. linearly indep.

(b) Is this set a basis for $\mathbb{P}_3$? Why or why not? If not, find a basis β of $\mathbb{P}_3$ which contains the given set.

No. Since $\dim \mathbb{P}_3 = 4$, we cannot have a basis of size 3.

$\beta = \{x, x^3+1, x^3-x, x^2\}$ works as a basis

3. (a) Might a system of 7 equations in 10 unknowns have three linearly independent solutions to the associated homogeneous system? Must it? Explain.

$$\begin{array}{cc} 7 & \begin{bmatrix} \quad & A & \quad \end{bmatrix} \\ & 10 \end{array}$$

$\text{rank } A \leq 7$

$\text{nullity } A \geq 3$

so yes, it must have at least 3 indep. solutions

(b) Might a system of 8 equations in 10 unknowns have three linearly independent solutions to the associated homogeneous system? Must it? Explain.

$$\begin{array}{cc} 8 & \begin{bmatrix} \quad & B & \quad \end{bmatrix} \\ & 10 \end{array}$$

$\text{rank } B \leq 8$

$\text{nullity } B \geq 2$

so it might but it need not.

(c) Might a system of 6 equations in 10 unknowns have *no more than* three linearly independent solutions to the associated homogeneous system? Explain.

$$\begin{array}{cc} 6 & \begin{bmatrix} \quad & C & \quad \end{bmatrix} \\ & 10 \end{array}$$

$\text{rank } C \leq 6$

$\text{nullity } C \geq 4$

so C must have at least 4 lin. indep solutions $\rightarrow$ so it cannot have only 3.

4. Give a careful definition of the following in terms of matrix multiplication:

λ is an eigenvalue of A.

λ is an eigenvalue of A if there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

5. Prove: If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

λ is an eigenvalue of A, $\exists\ \vec{v} \neq \vec{0}$ s.t. $A\vec{v} = \lambda v$

then $A^2\vec{v} = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda^2\vec{v}$

for some nonzero $\vec{v}$.

Thus λ^2 is an eigenvalue of A^2.

Alt: If $\det(A-\lambda I)=0$

then $\det\big((A-\lambda I)(A+\lambda I)\big)=0$

so $\det(A^2-\lambda^2 I)=0$

so λ^2 is an eigenvalue of A^2.