KEY

Math 300: Quiz the Last

This exam is closed book and closed notes. You may use a calculator for arithmetic only. You have until 5 minutes before the hour to finish the in-class portion. The take-home portion is due Monday at the beginning of class.

1. True or False. Give a brief justification in each case.

   (a) If we account for complex eigenvalues, any 2×2 matrix has exactly 2 (possibly equal) eigenvalues.

   True. $P(\lambda) = a\lambda^2 + b\lambda + c$ has two roots

   (b) If $2 - i$ is an eigenvalue of a matrix A, then so is $-2 - i$

   FALSE $\quad 2-i$ an eigenvalue $\to$ $2+i$ is an eigenvalue.

   (c) If $\mathbf{v}$ is orthogonal to a set of vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$, then $\mathbf{v}$ is orthogonal to any vector in $\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$.

   True $\quad \vec{v} \cdot \vec{w} = \vec{v} \cdot (c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_k \vec{w}_k) = c_1 \vec{v}_1 \cdot \vec{w}_1 + \cdots + c_k \vec{v} \cdot \vec{w}_k$
   $= 0 + 0 + \cdots + 0 = 0.$

   (d) The sum of two unit vectors is also a unit vector.

   FALSE

   $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}.$

   (e) For any vectors $\mathbf{x}, \mathbf{y}$ and any scalar λ, $(\lambda \mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

   FALSE $\quad (\lambda\vec{x}) \cdot \vec{y} = \lambda(\vec{x} \cdot \vec{y})$

2. Consider the matrix

$$A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$$

(a) Use the characteristic polynomial to prove that $\lambda = 2 - 3i$ is an eigenvalue of A.

$$\text{Char poly of } A = \begin{vmatrix} 3-\lambda & -5 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(1-\lambda) + 10$$

$$= 3 - 4\lambda + \lambda^2 + 10$$

$$= \lambda^2 - 4\lambda + 13 \qquad \text{ROOTS: } \lambda = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= 2 \pm \sqrt{-9}$$

$$= 2 \pm 3i$$

So $2-3i$ is an eigenvalue

(b) The corresponding eigenvector to λ is $\begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$. Use this to write $A = PCP^{-1}$, where C is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$P = \begin{bmatrix} \text{Re}\,\vec{v} & \text{Im}\,\vec{v} \end{bmatrix}$$

$$\lambda = 2 - 3i \rightarrow C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{6} \\ -\frac{2}{6} & \frac{1}{6} \end{bmatrix}$$

3. Prove that a 3×3 matrix with real entries has at least one real eigenvalue.

For a 3×3 matrix, $P(\lambda)\overset{=0}{}$ has the form

$$a\lambda^3+b\lambda^2+c\lambda+d=0$$

any cubic has at least 1 real root.

(complex roots come in conjugate pairs, thus an odd number of roots necessitates that at least one be real.

4. Let $\mathbf{y}=\begin{bmatrix}2\\3\\1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}-1\\-2\\2\end{bmatrix}$. Express $\mathbf{y}=\mathbf{x}+\mathbf{z}$, where $\mathbf{x}$ is a multiple of $\mathbf{v}$ and $\mathbf{z}$ is orthogonal to $\mathbf{v}$.

$$\vec{x} = \text{proj}_{\vec{v}}\vec{y} = \frac{\begin{bmatrix}2\\3\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\-2\\2\end{bmatrix}}{\begin{bmatrix}-1\\-2\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\-2\\2\end{bmatrix}}\begin{bmatrix}-1\\-2\\2\end{bmatrix} = \frac{-6}{9}\begin{bmatrix}-1\\-2\\2\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\-4/3\end{bmatrix}$$

$$\vec{z} = \vec{y}-\vec{x} = \begin{bmatrix}2\\3\\1\end{bmatrix} - \begin{bmatrix}2/3\\4/3\\-4/3\end{bmatrix} = \begin{bmatrix}4/3\\5/3\\7/3\end{bmatrix} \quad \text{so} \quad \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}4/3\\4/3\\-4/3\end{bmatrix} + \begin{bmatrix}4/3\\5/3\\7/3\end{bmatrix}$$

check: $$\begin{bmatrix}4/3\\5/3\\7/3\end{bmatrix}\cdot\begin{bmatrix}-1\\-2\\2\end{bmatrix} = \frac{-\cancel{4}}{3} - \frac{10}{3} + \frac{14}{3} = 0 \checkmark$$