Euler's Formula

Something every Math Major should know

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The Most Beautiful Formula...

We are asked to verify Euler's Formula

$$e^{i\pi} + 1 = 0$$

Some Series

Recall the following series expansions of transcendental functions from Calc 2

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots$
- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

Our formula for e^x holds for all values of x, real and imaginary. Thus...

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} +$$

Expanding with powers of i, we get the following

$$e^{i\theta} = 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \frac{(\theta)^4}{4!} +$$

Calculating $e^{i\theta}$

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The Different Parts

We break our expansion down into its real and imaginary parts...

$$\mathrm{Re}(e^{i\theta}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - = \cos(\theta)$$

$$\mathrm{Im}(e^{i\theta}) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - = \sin(\theta)$$

Thus, we conclude

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

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From the trigonometric representation of $e^{i\theta}$, it's a straightforward evaluation that yields Euler's Formula...

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i \cdot 0$$

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Deriving the Formula

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