A Rational Limit:
One more thing every Math Major should know

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A Useful Limit

Somewhere in your mathematical past, you've encountered the following formula for e:

'Definition' of e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

We seek to prove the formula using L'Hôspital's Rule and some trickery.

Proving the Limit

Rather than looking at

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

we will look at

$$\lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right)=\lim_{n\to\infty}n*\ln\left(1+\frac{1}{n}\right)$$

An Indeterminate Form

We note that as n gets large, our limit is of the form $\infty * 0$. We make a rearrangement to use L'Hôspital's Rule.

$$\lim_{n\to\infty} n * \ln\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to \frac{0}{0}$$

We now differentiate our numerator and denominator to obtain our limit.

$$\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$\xrightarrow{\frac{f'}{g'}} \frac{\frac{1}{1+\frac{1}{n}} * \frac{-1}{n^2}}{\frac{-1}{n^2}}$$

$$= \frac{1}{1+\frac{1}{n}}$$

Taking the Limit

Since

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1,$$

We have that

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

Hence,

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e$$