Sigma Connectivity - A Preliminary Report

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Outline

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   - Motivating Examples
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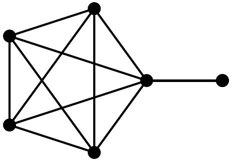
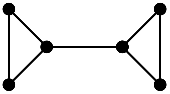
   - Other Examples
3. Total Σ Connectivity
   - Minimal Total Σ-Connectivity
   - Maximal Total Σ-Connectivity





Each example is 1-connected in the classical sense, but the first is ‘more’ one connected...

A more descriptive measure...

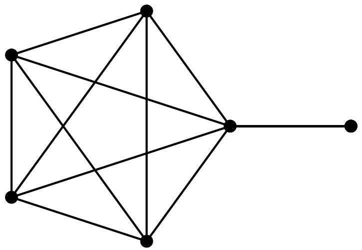
Can we devise a metric to distinguish between graphs that have ‘stronger’ k-connectivity than others?

Definition of Σ connectivity

Let G be a graph on n vertices. Define $s(G,k)$ to be the number of k element subsets of $V(G)$ such that the induced subgraph on $V(G) - S$ is connected. Then we define

$$\Sigma(G) = \sum_{k=0}^{n-2} \frac{s(G,k)}{\binom{n}{k}}$$

$K_5 + e$

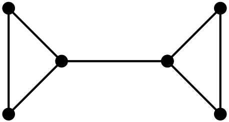


$$\Sigma(G) = 1 + \frac{5}{6} + \frac{1 + \binom{5}{2}}{\binom{6}{2}} + \frac{\binom{4}{1} + \binom{5}{3}}{\binom{6}{3}} + \frac{\binom{4}{2} + \binom{5}{4}}{\binom{6}{4}} = 4.0$$

P_6

$$\Sigma(G) = 1 + \frac{2}{6} + \frac{3}{\binom{6}{2}} + \frac{4}{\binom{6}{3}} + \frac{5}{\binom{6}{4}} = 2.06666\ldots$$

K_3 ‘join’ K_3



$$\Sigma(G) = 1 + \frac{4}{6} + \frac{\binom{4}{2}}{\binom{6}{2}} + \frac{2 + \binom{4}{3}}{\binom{6}{3}} + \frac{2 * \binom{3}{1} + \binom{4}{4}}{\binom{6}{4}} = 2.833333$$

A Few 'Easy' Results

If G is a graph with n vertices, then...

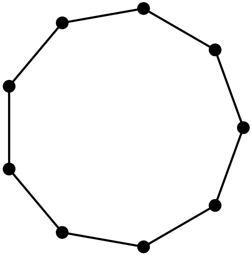
- $\Sigma(G) \geq \kappa(G)$
- $\Sigma(G) = n - 1$ iff $G = K_n$
- $\Sigma(G) = 0$ iff $G = K_n^c$
- $\Sigma(G) + \Sigma(G^c) \geq n - 1$

The Paths P_n

$$\Sigma(P_n) = 1 + \frac{2}{n} + \frac{3}{\binom{n}{2}} + \frac{4}{\binom{n}{3}} + \cdots + \frac{n-1}{\binom{n}{n-2}}$$

$$\lim_{n\to\infty} \Sigma(P_n) = 1$$

The Cycles C_n



$$\Sigma(C_n) = 1 + 1 + \frac{n}{\binom{n}{2}} + \frac{n}{\binom{n}{3}} + \cdots + \frac{n}{\binom{n}{n-2}}$$

$$\lim_{n\to\infty} \Sigma(C_n) = 2$$

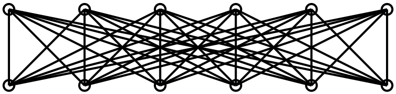
The Bipartite Graphs $K_{n,n}$

Recall that $K_{n,n}$ is n-connected.

A subset S of vertices of $K_{n,n}$ will disconnect the graph if and only if it contains all vertices in one part of the partition.

Thus, we calculate our Σ-connectivity by counting disconnecting sets and subtracting from the total.

The Graphs $K_{n,n}$

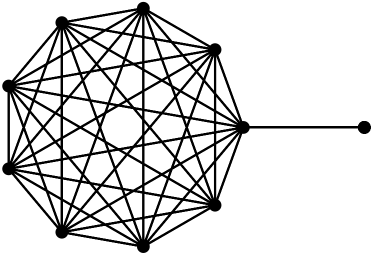


$$\Sigma(K_{n,n}) = 2n - 1 - \frac{2}{\binom{2n}{n}} - \frac{2*n}{\binom{2n}{n+1}} - \frac{2*\binom{n}{2}}{\binom{2n}{2n+2}} - \cdots - \frac{2*\binom{n}{i}}{\binom{2n}{n+i}} - \cdots - \frac{2*\binom{n}{n-2}}{\binom{2n}{2n-2}}$$

$$\Sigma(K_{n,n}) = 2n - 1 - \frac{n-1}{n+1}$$

$$\lim_{n\to\infty} \Sigma(K_{n,n}) = 2n - 2$$

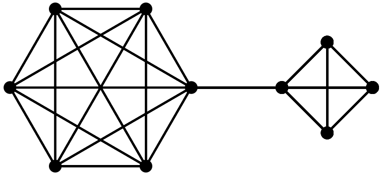
$K_n + e$



$$\Sigma = 1 + \frac{n}{n-1} + \frac{1+\binom{n}{2}}{\binom{n+1}{2}} + \frac{n-1+\binom{n}{3}}{\binom{n+1}{3}} + \cdots + \frac{\binom{n-1}{n-3}+\binom{n}{n-1}}{\binom{n+1}{n-1}}$$

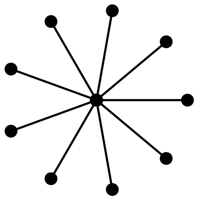
$$\lim_{n\to\infty} \Sigma(K_n + e) = \frac{5}{6}n$$

Barbell Graphs



$$\Sigma(B(m,n)) \to \frac{m+n}{3}$$

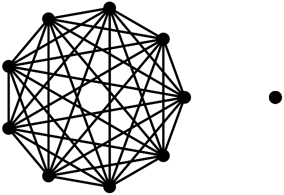
$K_{1,n}$



$$\Sigma(K_{1,n}) = 1 + \frac{n}{n+1} + \frac{\binom{n}{2}}{\binom{n+1}{2}} + \cdots + \frac{n}{\binom{n+1}{n-1}}$$

$$= \frac{n}{2} + 1 - \frac{1}{n+1}$$

A disconnected Example $K_n + v$



$$\Sigma(K_n + v) = 0 + \frac{1}{n+1} + \frac{n}{\binom{n+1}{2}} + \frac{\binom{n}{2}}{\binom{n+1}{3}} \cdots + \frac{\binom{n}{n-2}}{\binom{n+1}{n-1}}$$

$$= \frac{n}{2} - \frac{n}{n+1}$$

Definition

A basic graph theory result is that either G or G^c is connected.

Thus the calculation for $\Sigma(G) + \Sigma(G^c)$ will 'account' for each subset of vertices at least once.

We define the *Total Σ-Connectivity* of G as

$$T\Sigma(G) = \Sigma(G) + \Sigma(G^c)$$

Total Σ-connectivity

Our earlier calculations show that, if G has n vertices, then $T\Sigma(G) \geq n - 1$

We seek examples that achieve (and are furthest from!) this lower bound.

No P_4

We can show that a graph such that both G and G^c are connected must contain an induced P_4.

Thus, those subsets of vertices that contain an induced P_4 will contribute to ‘both’ parts of the Total Σ-Connectivity Calculation.

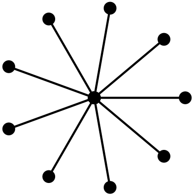
Minimal $T\Sigma(G)$

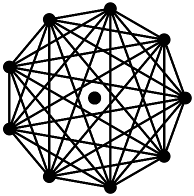
Theorem

For a graph G on n vertices, $T\Sigma(G) = n - 1$ if and only if neither G nor G^c contains an induced P_4.

Such graphs are known in the literature as *cographs*.

$K_{1,n}$ and $K_n + v$





$$T\Sigma(K_{1,n}) = \frac{n}{2} + 1 - \frac{1}{n+1} + \frac{n}{2} - \frac{n}{n+1}$$

$$T\Sigma(K_{1,n}) = n$$

Maximizing $T\Sigma(G)$

A theoretical upper bound for $T\Sigma(G)$ is $2n - 2$. Graphs approaching this bound should be 'rich' in induced P_4 subgraphs.

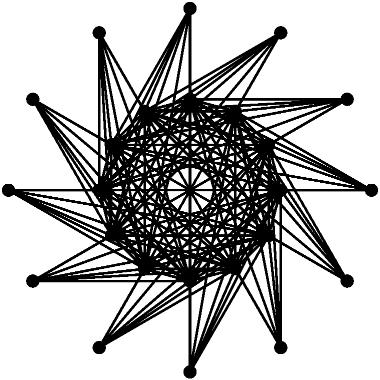
We have developed a family of graphs which we conjecture to approach this maximum in the limit.

Flower Graphs

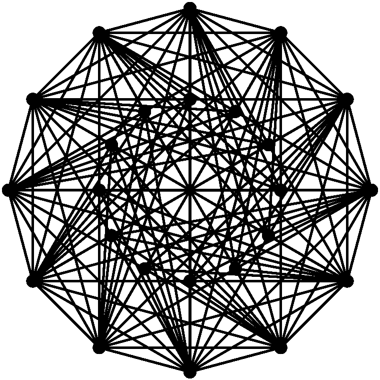
We define the Flower graphs F_n as a family of graphs on $4n$ vertices $V = \{v_0, v_1, \ldots v_{2n-1}, w_0, w_1, \ldots w_{2n-1}\}$ such that

- v_i is connected to v_j for all i, j
- w_i is connected to vertices $v_i, v_{i+1}, \ldots v_{i+n-1}$, where indices are computed mod $2n$.

F_{12}



F_{12}^c



Flower Graphs

Each Flower Graph is self-complementary, thus $T\Sigma(F_n) = 2\Sigma(F_n)$.

A set $S \subset V$ will disconnect F_n only if it contains n consecutive vertices v_i, and *not* the connecting w vertex.

An exact formula for $\Sigma(F_n)$

$$\Sigma(F_n) = \sum_{i=0}^{n-1} \sum_{k=i}^{i+2n} \binom{2n}{i}\binom{2n}{k-i} \Big/ \binom{4n}{k}$$

$$+ \sum_{i=n}^{2n-2} \sum_{j=n}^{i} \sum_{k=i+j-n+1}^{4n-2} 2n\binom{2n-j-2}{i-j}\binom{2n-(j-n+1)}{k-(i+j-n+1)} \Big/ \binom{4n}{k}$$

$$+ \sum_{k=2n-1}^{4n-2} 2n\binom{n}{k-(3n-1)} \Big/ \binom{4n}{k}$$

$$+ \sum_{i=n}^{2n-2} \left(\binom{2n}{i} - \sum_{j=n}^{i} 2n\binom{2n-j-2}{i-j} \right) \sum_{k=i}^{4n-2} \binom{2n}{k-i} \Big/ \binom{4n}{k}$$

Some Values for $\Sigma(F_n)$

n	$\Sigma(F_n)$	$\Sigma(F_n)/(4n)$
5	16.1221	0.8061
50	195.0875	0.9754
100	395.0168	0.9875
150	594.9928	0.9917

Other Directions

- Which graphs *maximize* Total Σ-connectivity?
- Can we prove our conjecture about the Flower Graphs?
- Are there ‘cleaner’ ways to compute $\Sigma(G)$ in general?

For more...

D. Corneil, H. Lerchs, and L. S. Burlingham
Complement Reducible Graphs
Discrete and Applied Mathematics, Vol. 3 (1981), p.163-174

D.H.Smith
Optimally Reliable Graphs for both Vertex and Edge Failures
Combinatorics, Probability and Computing, Vol. 2 (1993), p. 93-100