SYMBOLIC LOGIC

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Introduction: Logic and Language

1.1 What Is Logic?

It is easy to find answers to the question 'What is Logic?' According to Charles Peirce, 'Nearly a hundred definitions of it have been given'.[1] But Peirce goes on to write: 'It will, however, generally be conceded that its central problem is the classification of arguments, so that all those that are bad are thrown into one division, and those which are good into another . . .'.

The study of logic, then, is the study of the methods and principles used in distinguishing correct (good) from incorrect (bad) arguments. This definition is not intended to imply, of course, that only the student of logic can make the distinction. But the study of logic will *help* one to distinguish between correct and incorrect arguments, and it will do so in several ways. First of all, the proper study of logic will approach it as an art as well as a science, and the student will do exercises in all parts of the theory being learned. Here, as anywhere else, practice will help to make perfect. In the second place, the study of logic, especially symbolic logic, like the study of any other exact science, will tend to increase proficiency in reasoning. And finally, the study of logic will give students certain techniques for testing the validity of all arguments, including their own. This knowledge is of value because when mistakes are easily detected, they are less likely to be made.

Logic has frequently been defined as the science of reasoning. That definition, although it gives a clue to the nature of logic, is not quite accurate. Reasoning is that special kind of thinking called inferring, in which conclusions are drawn from premisses. As thinking, however, reasoning is not the special province of logic, but part of the psychologist's subject matter as well. Psychologists who examine the reasoning process find it to be extremely complex and highly emotional, consisting of awkward trial and error procedures that are illuminated by sudden—and sometimes apparently irrelevant—flashes of insight. These are all of importance to psychology. Logicians, however, are not interested in the actual process of reasoning, but rather with the correctness of

[1] Logic, in *Dictionary of Philosophy and Psychology*, James Mark Baldwin, ed., Macmillan

the completed reasoning process. Their question is always: Does the conclusion that is reached *follow* from the premisses used or assumed? If the premisses provide adequate grounds for accepting the conclusion, if asserting the premisses to be true warrants asserting the conclusion to be true also, then the reasoning is correct. Otherwise, the reasoning is incorrect. Logicians' methods and techniques have been developed primarily for the purpose of making this distinction clear. Logicians are interested in all reasoning, regardless of its subject matter, but only from this special point of view.

1.2 The Nature of Argument

Inferring is an activity in which one proposition is affirmed on the basis of one or more other propositions that are accepted as the starting point of the process. The logician is not concerned with the *process* of inference, but with the propositions that are the initial and end points of that process, and the relationships between them.

Propositions are either true or false, and in this they differ from questions, commands, and exclamations. Grammarians classify the linguistic formulations of propositions, questions, commands, and exclamations as declarative, interrogative, imperative, and exclamatory sentences, respectively. These are familiar notions. It is important to distinguish between declarative sentences and the propositions they may be uttered to assert. A declarative sentence is always part of a language, the language in which it is spoken or written, whereas propositions are not peculiar to any of the languages in which they may be expressed. Another difference between declarative sentences and propositions is that the same sentence may be uttered in different contexts to assert different propositions. (For example, the sentence 'I am hungry' may be uttered by different persons to make different assertions.)

The same sort of distinction can be drawn between sentences and *statements*. The same statement can be made using different words, and the same sentence can be uttered in different contexts to make different statements. The terms 'proposition' and 'statement' are not exact synonyms, but in the writings of logicians they are used in much the same sense. In this book, both terms will be used. In the following chapters, we shall also use the term 'statement' (especially in Chapters 2 and 3) and the term 'proposition' (especially in Chapters 4 and 5) to refer to the sentences in which statements and propositions are expressed. In each case the context should make clear what is meant.

Corresponding to every possible inference is an *argument*, and it is with these arguments that logic is chiefly concerned. An argument may be defined as any group of propositions or statements, of which one is claimed to follow from the others, which are alleged to provide grounds for the truth of that one. In ordinary usage, the word 'argument' also has other meanings, but in logic it has the technical sense explained. In the following chapters we shall use the word 'argument' also in a derivative sense to refer to any sentence or collection of

sentences in which an argument is formulated or expressed. When we do, we shall be presupposing that the context is sufficiently clear to ensure that unique statements are made or unique propositions are asserted by the utterance of those sentences.

Every argument has a structure, in the analysis of which the terms 'premiss' and 'conclusion' are usually employed. The *conclusion* of an argument is that proposition which is affirmed on the basis of the other propositions of the argument. These other propositions, which are affirmed as providing grounds or reasons for accepting the conclusion, are the *premisses* of that argument.

We note that 'premiss' and 'conclusion' are relative terms, in the sense that the same proposition can be a premiss in one argument and a conclusion in another. Thus the proposition *All humans are mortal* is premiss in the argument

> All humans are mortal.
> Socrates is human.
> Therefore, Socrates is mortal.

and conclusion in the argument

> All animals are mortal.
> All humans are animals.
> Therefore, all humans are mortal.

Any proposition can be either a premiss or a conclusion, depending upon its context. It is a premiss when it occurs in an argument in which it is assumed for the sake of proving some other proposition. And it is a conclusion when it occurs in an argument that is claimed to prove it on the basis of other propositions that are assumed.

It is customary to distinguish between *deductive* and *inductive* arguments. All arguments involve the claim that their premisses provide some grounds for the truth of their conclusions, but only a *deductive* argument involves the claim that its premisses provide *absolutely conclusive* grounds. The technical terms 'valid' and 'invalid' are used in place of 'correct' and 'incorrect' in characterizing deductive arguments. A deductive argument is *valid* when its premisses and conclusion are so related that it is absolutely impossible for the premisses to be true unless the conclusion is true also. The task of deductive logic is to clarify the nature of the relationship that holds between premisses and conclusion in a valid argument, and to provide techniques for discriminating valid from invalid arguments.

Inductive arguments involve the claim only that their premisses provide *some* grounds for their conclusions. Neither the term 'valid' nor its opposite 'invalid' is properly applied to inductive arguments. Inductive arguments differ among themselves in the degree of likelihood or probability that their premisses confer upon their conclusions. Inductive arguments are tive logic. In this book, however, we shall be concerned only with deductive arguments, and shall use the word 'argument' to refer to deductive arguments exclusively.

1.3 Truth and Validity

Truth and falsehood characterize propositions or statements and may also be said to characterize the declarative sentences in which they are formulated. Arguments, however, are not properly characterized as being either true or false but as valid or invalid.[2] There is a connection between the validity or invalidity of an argument and the truth or falsehood of its premisses and conclusion, but the connection is by no means a simple one.

Some valid arguments contain true propositions only, as, for example,

All bats are mammals.
All mammals have lungs.
Therefore, all bats have lungs.

An argument may contain false propositions exclusively and still be valid, as, for example,

All trout are mammals.
All mammals have wings.
Therefore, all trout have wings.

This argument is valid because *if* its premisses were true, its conclusion would have to be true also, even though, in fact, they are all false. These two examples show that although some valid arguments have true conclusions, not all of them do. The validity of an argument does not, therefore, guarantee the truth of its conclusion.

When we consider the argument

If I am President, then I am famous.
I am not President.
Therefore, I am not famous.

we can see that although both premisses and conclusion are true, the argument is invalid. Its invalidity becomes obvious when it is compared with another argument of the same form:

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[2] Some logicians use the term 'valid' to characterize statements that are *logically true*. We shall adopt that usage in Chapter 10, Section 10.6. Until then, however, we apply the terms 'valid' and 'invalid' to arguments exclusively.

If Rockefeller is President, then Rockefeller is famous.
Rockefeller is not President.
Therefore, Rockefeller is not famous.

This argument is clearly invalid because its premisses are true but its conclusion is false. The two latter examples show that although some invalid arguments have false conclusions, not all of them do. The falsehood of its conclusion does not guarantee the invalidity of an argument. But the falsehood of its conclusion does guarantee that *either* the argument is invalid *or* at least one of its premisses is false.

An argument must satisfy two conditions to establish the truth of its conclusion. It must be valid, and all of its premisses must be true. Such an argument is termed 'sound'. To determine the truth or falsehood of premisses is the task of scientific inquiry in general, since premisses may deal with any subject matter at all. But determining the validity or invalidity of arguments is the special province of deductive logic. The logician is interested in the question of validity even for arguments that might be unsound because their premisses might happen to be false.

A question might be raised about the legitimacy of that interest. It might be suggested that logicians should confine their attention to arguments that have true premisses only. It is often necessary, however, to depend upon the validity of arguments whose premisses are not known to be true. Scientists test their theories by deducing from them conclusions that predict the behavior of observable phenomena in the laboratory or observatory. The conclusion is then tested directly by observation of experimental data and if it is true, the results confirm the theory from which the conclusion was deduced. If the conclusion is false, the results disconfirm or refute the theory. In either case, the scientist is vitally interested in the validity of the argument by which the testable conclusion is deduced from the theory being investigated, for if that argument is invalid, his whole procedure is without point. Although an oversimplification of scientific method, our example shows that questions of validity are important even for arguments whose premisses are not true.

1.4 Symbolic Logic

It has been explained that logic is concerned with arguments and that arguments contain propositions or statements as their premisses and conclusions. These premisses and conclusions are not linguistic entities, such as declarative sentences, but are, rather, what declarative sentences are typically uttered to assert. The communication of propositions and arguments, however, requires the use of language, and this complicates our problem. Arguments formulated in English or any other natural language are often difficult to appraise because of the vague and equivocal nature of the words in which they are expressed, the ambiguity of their construction, the misleading idioms they may contain, and their pleasing but deceptive metaphorical style. The resolution of these difficulties is not the central problem for the logician, however, for even when they are resolved, the problem of deciding the validity or invalidity of the argument still remains.

To avoid the peripheral difficulties connected with ordinary language, workers in the various sciences have developed specialized technical vocabularies. The scientist economizes the space and time required for writing his reports and theories by adopting special symbols to express ideas that would otherwise require a long sequence of familiar words to formulate. This has the further advantage of reducing the amount of attention needed, for when a sentence or equation grows too long, its meaning is more difficult to grasp. The introduction of the exponent symbol in mathematics permits the expression of the equation

$$A \times A \times A \times A \times A \times A \times A \times A \times A \times A \times A \times A = B \times B \times B \times B \times B \times B \times B$$

more briefly and intelligibly as

$$A^{12} = B^{7}$$

A like advantage has been obtained by the use of graphic formulas in organic chemistry. The language of every advanced science has been enriched by similar symbolic innovations.

A special technical notation has been developed for logic as well. Aristotle made use of certain abbreviations to facilitate his own investigations. Modern symbolic logic augmented this base by the introduction of many more special symbols. The difference between the old and the new logic is one of degree rather than of kind, but the difference in degree is tremendous. Modern symbolic logic has become immeasurably more powerful a tool for analysis and deduction through the development of its own technical language. The special symbols of modern logic permit us to exhibit with greater clarity the logical structures of arguments that may be obscured by formulation in ordinary language. It is easier to divide arguments into the valid and the invalid when they are expressed in a special symbolic language, for with symbols the peripheral problems of vagueness, ambiguity, idiom, metaphor, and amphiboly do not arise. The introduction and use of special symbols serve not only to facilitate the appraisal of arguments, but also to clarify the nature of deductive inference.

The logician's special symbols are much better adapted to the actual drawing of inferences than is ordinary language. Their superiority in this respect is comparable to that of Arabic over the older Roman numerals for purposes of computation. It is easy to multiply 148 by 47, but very difficult to compute the product of CXLVIII and XLVII. Similarly, the drawing of

inferences and the evaluation of arguments is greatly facilitated by the adoption of a special logical notation. To quote Alfred North Whitehead, an important contributor to the advance of symbolic logic:

> . . . by the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain.[3]

Although this book treats symbolic logic systematically rather than historically, a few historical remarks may be appropriate at this point. Since the 1840s, symbolic logic has developed along two different historical paths. One of them began with the English mathematician George Boole (1815–1864). Boole applied algebraic notations and methods first to symbolize and then to validate arguments of the kind studied by Aristotle in the fourth century B.C. This route may be characterized as an effort to apply mathematical notations and methods to traditional, nonmathematical kinds of arguments. The other path began with the independent efforts of the English mathematician Augustus De Morgan (1806–1871) and the American scientist and philosopher Charles Peirce (1839–1914) to devise a very precise notation for relational arguments. The earlier logic had largely ignored this type of argument, which, nevertheless, plays a central role in mathematics. This historical path may be characterized as an effort to create a new quasi-mathematical kind of logical notation and analytical technique for use in mathematical derivations and demonstrations.

These two historical paths coalesced in the brilliant works of the German mathematician and philosopher Gottlob Frege (1848–1925), the Italian mathematician Guiseppe Peano (1858–1932), and the English philosophers Alfred North Whitehead (1861–1947) and Bertrand Russell (1872–1970), whose *Principia Mathematica* was an important landmark in the history of symbolic logic. Some of Boole's contributions are reported in the first two sections of Chapter 7 and in Appendix B. The contributions of the others have become so thoroughly incorporated into modern symbolic logic that only occasional references to their more distinctive ideas are appropriate.

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[3] A. N. Whitehead, *An Introduction to Mathematics*, Oxford University Press, Oxford, England, 1911

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Arguments Containing Compound Statements

2.1 Simple and Compound Statements

All statements can be divided into two kinds, simple and compound. A *simple* statement is one that does not contain any other statement as a component part, whereas every *compound* statement does contain another statement as a component part. For example, 'Atmospheric testing of nuclear weapons will be discontinued or this planet will become uninhabitable' is a compound statement that contains, as its components, the two simple statements 'Atmospheric testing of nuclear weapons will be discontinued' and 'this planet will become uninhabitable'. The component parts of a compound statement may themselves be compound, of course.

As the term 'component' is used in logic, not every statement that is part of a larger statement is a component of the larger statement. For example, the last six words of the statement, 'The third wife of Bertrand Russell was a beautiful girl', can be regarded as a statement in its own right. But it is not a *component* part or a *component* of the larger statement containing it. For a part of a statement to be a component of a larger statement, two conditions must be satisfied. First, the part must be a statement in its own right; and second, if the part is replaced in the larger statement by any other statement, the result of that replacement must be meaningful. Although the first condition is satisfied in the example given, the second is not. For if the part 'Bertrand Russell was a beautiful girl' is replaced by the statement 'Where there's smoke, there's fire', the result of that replacement is arrant nonsense.[1]

The statement 'Roses are red and violets are blue' is a *conjunction*, a compound statement formed by inserting the word 'and' between two statements. Two statements so combined are called *conjuncts*. The word 'and' has other uses, however, as in the statement 'Castor and Pollux were twins', which is not compound, but a simple statement asserting a relationship. We introduce the dot '·' as a special symbol for combining statements conjunctively. Using

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[1] This account of compound and component statements has been suggested by Professor C. Mason Myers of Northern Illinois University, Professor Alex Blum of Bar-Ilan University, and Professor James A. Martin of the University of Wyoming. See Martin's 'How Not to Define Truth-Functionality,' *Logique et Analyse*, 1970, 14^e Annee, N° 52, pp. 476–482.

this notation, the preceding conjunction is written 'Roses are red·violets are blue'. Where p and q are any two statements whatever, their conjunction is written $p \cdot q$.

Every statement is either true or false, so we can speak of the *truth value* of a statement, where the truth value of a true statement is *true* and the truth value of a false statement is *false*. There are two broad categories into which compound statements can be divided, according to whether or not there is anything other than the truth values of its component statements that determines the truth value of the compound statement. The truth value of the conjunction of two statements is completely determined by the truth value of its conjuncts. A conjunction is true if both its conjuncts are true, but false otherwise. For this reason a conjunction is a *truth-functional* compound statement, and its conjuncts are *truth-functional* components of it.

Not every compound statement is truth-functional. For example, the truth-value of the compound statement 'Percival Lowell believed that Mars is inhabited' is not in any way determined by the truth value of its component simple statement 'Mars is inhabited'. This is so because it could be true that Percival Lowell believed that Mars is inhabited, regardless of whether it is inhabited or not. A person can believe one truth without believing another, and a person can believe one falsehood without believing all of them. So the component 'Mars is inhabited' is not a truth-functional component of the compound statement 'Percival Lowell believed that Mars is inhabited,' and the latter statement is not a truth-functional compound statement.

We define an occurrence of a component of a compound statement to be a *truth-functional component* of that compound statement provided that: If that occurrence of the component is replaced in the compound (or in any component of the compound statement which contains the component in question) by different statements that have the same truth value as each other, the different compound statements produced by these replacements will also have the same truth values as each other. And now a compound statement is defined to be a *truth-functional compound statement,* if all occurrences in it of its components are truth-functional components of the statement.[2]

The only compound statements we shall consider here will be truth-functionally compound statements. Therefore, in the rest of this book we shall use the term "simple statement" to refer to any statement that is not truth-functionally compound.

Since conjunctions are truth-functionally compound statements, our dot symbol is a truth-functional connective. Given any two statements p and q there are just four possible sets of truth values they can have, and in every case the truth value of their conjunction $p \cdot q$ is uniquely determined. The four possible cases can be exhibited as follows:

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[2] Compare this definition with the one proposed by Professor David H. Sanford in his 'What Is a Truth Functional Component?' *Logique et Analyse*, 1970, 14^e Annee, N° 52, pp. 483–486.

in case p is true and q is true, $p \cdot q$ is true;
in case p is true and q is false, $p \cdot q$ is false;
in case p is false and q is true, $p \cdot q$ is false;
in case p is false and q is false, $p \cdot q$ is false.

Representing the truth values true and false by the capital letters ‘**T**’ and ‘**F**’, respectively, the way in which the truth value of a conjunction is determined by the truth values of its conjuncts can be displayed more briefly by means of a *truth table* as follows:

p	q	$p \cdot q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Since it specifies the truth value of $p \cdot q$ in every possible case, this truth table can be taken as *defining* the dot symbol. Other English words such as ‘moreover’, ‘furthermore’, ‘but’, ‘yet’, ‘still’, ‘however’, ‘also’, ‘nevertheless’, ‘although’, and so forth, and even the comma and the semicolon, are also used to conjoin two statements into a single compound one, and all of them can be indifferently translated into the dot symbol, so far as truth values are concerned.

The statement ‘It is not the case that lead is heavier than gold’ is also compound, being the *negation* (or *denial* or *contradictory*) of its single component statement ‘lead is heavier than gold’. We introduce the symbol ‘$\sim$’, called a *curl* (or a *tilde*) to symbolize negation. There are often alternative formulations in English of the negation of a given statement. Thus where L symbolizes the statement ‘lead is heavier than gold’, the different statements ‘it is not the case that lead is heavier than gold’, ‘it is false that lead is heavier than gold’, ‘it is not true that lead is heavier than gold’, ‘lead is not heavier than gold’ are all indifferently symbolized as $\sim L$. More generally, where p is any statement whatever, its negation is written $\sim p$. Since the negation of a true statement is false and the negation of a false statement is true, we can take the following truth table as defining the curl symbol:

p	$\sim p$
T	**F**
F	**T**

When two statements are combined disjunctively by inserting the word ‘or’ between them, the resulting compound statement is a *disjunction* (or *alternation*), and the two statements so combined are called *disjuncts* (or *alternatives*). The word ‘or’ has two different senses, one of which is clearly intended in the

statement ‘Premiums will be waived in the event of sickness or unemployment’. The intention here is obviously that premiums are waived not only for sick persons and for unemployed persons, but also for persons who are both sick *and* unemployed. This sense of the word ‘or’ is called *weak* or *inclusive*. Where precision is at a premium, as in contracts and other legal documents, this sense is made explicit by use of the phrase ‘and/or’.

A different sense of ‘or’ is intended when a restaurant lists ‘tea or coffee’ on its table d’hôte menu, meaning that for the stated price of the meal the customer can have one or the other, but *not both*. This second sense of ‘or’ is called *strong* or *exclusive*. Where precision is at a premium and the exclusive sense of ‘or’ is intended, the phrase ‘but not both’ is often added.

A disjunction which uses the inclusive ‘or’ asserts that *at least one disjunct is true*, whereas one which uses the exclusive ‘or’ asserts that *at least one disjunct is true but not both are true*. The *partial common meaning* that at least one disjunct is true, is the whole meaning of an inclusive disjunction and a part of the meaning of an exclusive disjunction.

In Latin, the word ‘vel’ expresses the inclusive sense of the word ‘or’ and the word ‘aut’ expresses the exclusive sense. It is customary to use the first letter of ‘vel’ to symbolize ‘or’ in its inclusive sense. Where p and q are any two statements whatever, their weak or inclusive disjunction is written $p \vee q$. The symbol ‘v’, called a *wedge* (or a *vee*), is a truth-functional connective, and is defined by the following truth table:

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

An obviously valid argument containing a disjunction is the following Disjunctive Syllogism:

> The United Nations will become more responsible or there will be a third world war.
> The United Nations will not become more responsible.
> Therefore, there will be a third world war.

It is evident that a Disjunctive Syllogism is valid on *either* interpretation of the word ‘or’, that is, regardless of whether its first premiss asserts an inclusive or exclusive disjunction. It is usually difficult, and sometimes impossible, to discover which sense of the word ‘or’ is intended in a disjunction. But the typical valid argument that has a disjunction for a premiss is, like the Disjunctive Syllogism, valid on either interpretation of the word ‘or’. Hence we effect a simplification by translating any occurrence of the word ‘or’ into the logical symbol ‘v’, *regardless of which* where it is explicitly stated that the disjunction is exclusive, by use of the added phrase 'but not both', for example, we do have the symbolic apparatus for expressing that sense, as will be explained below.

The use of parentheses, brackets, and braces for punctuating mathematical expressions is familiar. In the absence of a special convention, no number is uniquely denoted by the expression '$6 + 9 \div 3$', although when punctuation makes clear how its constituents are to be grouped, it denotes either 5 or 9. Punctuation is needed to resolve ambiguity in the language of symbolic logic too, since compound statements may themselves be combined to produce more complicated compounds. Ambiguity is present in $p \cdot q \vee r$, which could be either the conjunction of p with $q \vee r$, or else the disjunction of $p \cdot q$ with r. These two different senses are unambiguously given by different punctuations: $p \cdot (q \vee r)$ and $(p \cdot q) \vee r$. In case p and q are both false and r is true, the first punctuated expression is false (since its first conjunct is false), but the second punctuated expression is true (since its second disjunct is true). Here a difference in punctuation makes all the difference between truth and falsehood. In symbolic logic, as in mathematics, we use parentheses, brackets, and braces for punctuation. To cut down on the number of punctuation marks required, however, we establish the symbolic convention that in any expression the curl will apply to the smallest component that the punctuation permits. Thus the ambiguity of $\sim p \vee q$, which might mean either $(\sim p) \vee q$ or $\sim (p \vee q)$, is resolved, by our convention, to mean the first of these. The curl can, and therefore by our convention *does*, apply to the first component p rather than to the larger expression $p \vee q$.

The word 'either' has a variety of different uses in English. It has conjunctive force in 'The Disjunctive Syllogism is valid on either interpretation of the word 'or'.' It frequently serves merely to introduce the first disjunct in a disjunction, as in 'Either the United Nations will become more responsible or there will be a third world war'. Perhaps the most useful function of the word 'either' is to punctuate some compound statements. Thus the sentence

> More stringent antipollution measures will be enacted and the laws will be strictly enforced or the quality of life will be degraded still further.

can have its ambiguity resolved in one direction by placing the word 'either' at its beginning, or in the other direction by inserting the word 'either' right after the word 'and'. Such punctuation is effected in our symbolic language by parentheses. The ambiguous formula $p \cdot q \vee r$ discussed in the preceding paragraph corresponds to the ambiguous sentence considered in this one. The two different punctuations of the formula correspond to the two different punctuations of the sentence effected by the two different insertions of the word 'either'.

Not all conjunctions are formulated by explicitly placing the word 'and' between complete sentences, as in 'Charlie's neat and Charlie's sweet'. Indeed, the latter would more naturally be expressed as 'Charlie's neat and sweet'. The

familiar 'Jack and Jill went up the hill' is the more natural way of expressing the conjunction 'Jack went up the hill and Jill went up the hill'. It is the same with disjunctions: 'Either Alice or Betty will be elected' expresses more briefly the proposition alternatively formulated as 'Either Alice will be elected or Betty will be elected'; and 'Charlene will be either secretary or treasurer' expresses somewhat more briefly the same proposition as 'Either Charlene will be secretary or Charlene will be treasurer'.

The negation of a disjunction is often expressed by using the phrase 'neither-nor'. Thus the disjunction 'Either Alice or Betty will be elected' is denied by the statement 'Neither Alice nor Betty will be elected'. The disjunction would be symbolized as $A \vee B$ and its negation as either $\sim (A \vee B)$ or as $(\sim A) \cdot (\sim B)$. (The logical equivalence of these two formulas will be discussed in Section 2.4.) To deny that at least one of two statements is true is to assert that both of the two statements are false.

The word 'both' serves various functions. One is simply a matter of emphasis. To say 'Both Jack and Jill went up the hill' is only to emphasize that the two of them did what they are asserted to have done by saying 'Jack and Jill went up the hill'. A more useful function of the word 'both' is punctuational, like that of 'either'. 'Both . . . and –•– are not ---' is used to make the same statement as 'Neither . . . nor –•– is ---. In such sentences, the *order* of the words 'both' and 'not' is very significant. There is a great difference between

> Alice and Betty will not both be elected.

and

> Alice and Betty will both not be elected.

The former would be symbolized as $\sim (A \cdot B)$, the latter as $(\sim A) \cdot (\sim B)$.

Finally, it should be remarked that the word 'unless' can also be used in expressing the disjunction of two statements. Thus 'Our resources will soon be exhausted unless more recycling of materials is effected' and 'Unless more recycling of materials is effected our resources will soon be exhausted' can equally well be expressed as 'Either more recycling of materials is effected or our resources will soon be exhausted' and symbolized as $M \vee E$.

Since an exclusive disjunction asserts that at least one of its disjuncts is true but that they are not both true, we can symbolize the exclusive disjunction of any two statements p and q quite simply as $(p \vee q) \cdot \sim (p \cdot q)$. Thus we are able to symbolize conjunctions, negations, and both inclusive and exclusive disjunctions. Any compound statement that is built up out of simple statements by repeated use of truth-functional connectives will have its truth value completely determined by the truth values of those simple statements. For example, if A and B are true statements and X and Y are false, the truth value of the compound statement $\sim [(\sim A \vee X) \vee \sim (B \cdot Y)]$ can be discovered as foll false also, the disjunction $(\sim A \vee X)$ is false. Since Y is false, the conjunction $(B \cdot Y)$ is false, and so its negation $\sim(B \cdot Y)$ is true. Hence the disjunction $(\sim A \vee X) \vee \sim(B \cdot Y)$ is true, and its negation, which is the given statement, is false. Such a stepwise procedure, beginning with the inmost components, always permits us to determine the truth value of a truth-functionally compound statement given the truth values of its component simple statements.

EXERCISES[3]

I. If A and B are true statements and X and Y are false statements, which of the following compound statements are true?

***1.** $\sim(A \vee X)$
2. $\sim A \vee \sim X$
3. $\sim B \cdot \sim Y$
4. $\sim(B \cdot Y)$
***5.** $A \vee (X \cdot Y)$
6. $(A \vee X) \cdot Y$
7. $(A \vee B) \cdot (X \vee Y)$
8. $(A \cdot B) \vee (X \cdot Y)$
9. $(A \cdot X) \vee (B \cdot Y)$
***10.** $A \cdot [X \vee (B \cdot Y)]$
11. $A \vee [X \cdot (B \vee Y)]$
12. $X \vee [A \cdot (Y \vee B)]$
13. $\sim\{\sim[\sim(A \cdot \sim X) \cdot \sim A] \cdot \sim X\}$
14. $\sim\{\sim[\sim(A \cdot \sim B) \cdot \sim A] \cdot \sim A\}$
***15.** $[(A \cdot X) \vee \sim B] \cdot \sim[(A \cdot X) \vee \sim B]$
16. $[(X \cdot A) \vee \sim Y] \vee \sim[(X \cdot A) \vee \sim Y]$
17. $[A \cdot (X \vee Y)] \vee \sim[(A \cdot X) \vee (A \cdot Y)]$
18. $[X \vee (A \cdot Y)] \vee \sim[(X \vee A) \cdot (X \vee Y)]$
19. $[X \cdot (A \vee B)] \vee \sim[(X \vee A) \cdot (X \vee B)]$
20. $[X \vee (A \cdot Y)] \vee \sim[(X \vee A) \vee (X \vee Y)]$

II. If A and B are known to be true, and X and Y are known to be false, but the truth values of P and Q are not known, of which of the following statements can you determine the truth value?

***1.** $P \cdot X$
2. $B \vee Q$
3. $\sim A \cdot P$
4. $Q \vee \sim Q$
***5.** $P \vee \sim Y$
6. $P \cdot \sim P$
7. $\sim P \vee (\sim A \vee P)$
8. $\sim P \vee (\sim X \vee P)$
9. $\sim P \vee (\sim Q \vee P)$
***10.** $Q \vee \sim(P \cdot Q)$
11. $(P \cdot Q) \vee (\sim Q \vee \sim P)$
12. $(P \vee Q) \vee (\sim P \cdot \sim Q)$
13. $(P \vee Q) \cdot (\sim B \vee Y)$
14. $(P \cdot Q) \vee \sim(Q \cdot P)$
***15.** $(P \cdot Q) \cdot (\sim A \vee X)$
16. $\sim(P \cdot Q) \vee P$
17. $(\sim P \vee Q) \vee (P \vee \sim Q)$
18. $[P \vee (Q \vee X)] \cdot \sim[(P \vee Q) \vee X]$
19. $[P \vee (Q \cdot A)] \cdot \sim[(P \vee Q) \cdot (P \vee A)]$
20. $[P \cdot (Q \vee X)] \vee \sim[(P \cdot Q) \vee (P \cdot X)]$

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[3] Solutions to starred exercises will be found on pages 355–380.

III. Using the letters A, B, C, and D to abbreviate the simple statements: 'Atlanta wins their conference championship', 'Baltimore wins their conference championship', 'Chicago wins the superbowl', and 'Dallas wins the superbowl', symbolize the following:

***1.** Either Atlanta wins their conference championship and Baltimore wins their conference championship or Chicago wins the superbowl.
2. Atlanta wins their conference championship and either Baltimore wins their conference championship or Dallas does not win the superbowl.
3. Atlanta and Baltimore will not both win their conference championships but Chicago and Dallas will both not win the superbowl.
4. Either Atlanta or Baltimore will win their conference championships but neither Chicago nor Dallas will win the superbowl.
***5.** Either Chicago or Dallas will win the superbowl but they will not both win the superbowl.
6. Chicago will win the superbowl unless Atlanta wins their conference championship.
7. It is not the case that neither Atlanta nor Baltimore wins their conference championship.
8. Either Chicago or Dallas will fail to win the superbowl.
9. Either Chicago or Dallas will win the superbowl unless both Atlanta and Baltimore win their conference championships.
10. Either Chicago will win the superbowl and Dallas will not win the superbowl or both Atlanta and Baltimore will win their conference championships.

IV. Using capital letters to abbreviate simple statements, symbolize the following:

***1.** The words of his mouth were smoother than butter, but war was in his heart. (Psalm 55:21)
2. Promotion cometh neither from the east, nor from the west, nor yet from the south. (Psalm 75:6)
3. As for man, his days are as grass: as a flower of the field, so he flourisheth. (Psalm 103:15)
4. Wine is a mocker, strong drink is raging. (Proverbs 20:1)
***5.** God hath made man upright; but they have sought out many inventions (Ecclesiastes 7:29)
6. The race is *not* to the swift, nor the battle to the strong . . . (Ecclesiastes 9:11)
7. Love is strong as death; jealousy is cruel as the grave. (The Song of Solomon 8:6)
8. A bruised reed shall he not break, and the smoking flax shall he not quench. (Isaiah 42:3)
9. Saul and Jonathan were lovely and pleasant in their lives . . . (2 Samuel 1:23)
10. His eye was not dim, nor his natural force abated. (Deuteronomy 34:7)
11. The voice is Jacob's voice, but the hands are the hands of Esau. (Genesis 27:22)
12. He shall return no more to his house, neither shall his place know him any more. (Job 7:10)

2.2 Conditional Statements

The compound statement 'If the train is late, then we shall miss our connection' is a *conditional* (or a *hypothetical*, an *implication*, or an *implicative statement*). The component between the 'if' and the 'then' is called the *antecedent* (or the *implicans* or *protasis*), and the component that follows the 'then' is the *consequent* (or the *implicate* or *apodosis*). A conditional does not state either that its antecedent is true or that its consequent is true; it says only that *if* its antecedent is true, then its consequent is true also, that is, its antecedent *implies* its consequent. The key to the meaning of a conditional is the relation of *implication* that is asserted to hold between its antecedent and its consequent, in that order.

If we examine a number of different conditionals, we can see that there are different kinds of implications which they may express. In the conditional 'If all cats like liver and Dinah is a cat, then Dinah likes liver', the consequent follows *logically* from its antecedent. On the other hand, in the conditional 'If the figure is a triangle, then the figure has three sides', the consequent follows from the antecedent by the very *definition* of the word 'triangle'. The truth of the conditional 'If this piece of gold is placed in *aqua regia*, then this piece of gold dissolves' is not a matter of either logic or definition. The connection asserted here is *causal* and must be discovered empirically. These examples show that there are different kinds of implications that constitute different senses of the 'if-then' phrase. Having noted these differences, we now seek to find some identifiable common meaning, some partial meaning that is common to these different types of conditionals.

Our discussion of 'if-then' will parallel our previous discussion of the word 'or'. First, we pointed out two different senses of that word. Second, we noted that there was a common partial meaning: That *at least one disjunct is true* was seen to be involved in both the inclusive and the exclusive 'or'. Third, we introduced the special symbol 'v' to represent this common partial meaning (which was the whole meaning of 'or' in its inclusive sense). Fourth, we observed that because arguments like the Disjunctive Syllogism are valid on either interpretation of the word 'or', symbolizing *any* occurrence of the word 'or' by the wedge symbol, preserves the validity of such arguments. And because we are interested in arguments from the point of view of determining their validity, this translation of the word 'or' into 'v', which may abstract from or ignore part of its meaning in some cases, is wholly adequate for our present purposes.

A common partial meaning of these different kinds of conditional statements emerges when we ask what circumstances would suffice to establish the *falsehood* of a conditional. Under what circumstances would we agree that the conditional 'If this piece of gold is placed in this solution, then this piece of gold dissolves' is false? Clearly, the statement is false in the case that this piece

with a true antecedent and a false consequent must be false. Hence any conditional *if* p, *then* q is known to be false in case the conjunction $p \cdot {\sim} q$ is known to be true, that is, in case the antecedent of the conditional is true and its consequent false. For the conditional to be true, the indicated conjunction must be false, which means that the negation of that conjunction must be true. In other words, for any conditional *if* p, *then* q to be true, ${\sim}(p \cdot {\sim} q)$, the negation of the conjunction of its antecedent with the negation of its consequent, must be true also. We may, then, regard the latter as a *part* of the meaning of the former.

We introduce the new symbol '$\supset$', called a *horseshoe*, to represent the partial meaning that is common to all conditional statements. We define '$p \supset q$' as an abbreviation for '${\sim}(p \cdot {\sim} q)$'. The horseshoe is a truth-functional connective, whose exact significance is indicated by the following truth table:

p	q	${\sim} q$	$p \cdot {\sim} q$	${\sim}(p \cdot {\sim} q)$	$p \supset q$
T	**T**	**F**	**F**	**T**	**T**
T	**F**	**T**	**T**	**F**	**F**
F	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**F**	**T**	**T**

Here, the first two columns represent all possible truth values for the component statements p and q, and the third, fourth, and fifth represent successive stages in determining the truth value of the compound statement ${\sim}(p \cdot {\sim} q)$ in each case. The sixth column is identically the same as the fifth, since the formulas that head them are defined to express the same proposition. The horseshoe symbol must not be thought of as representing *the* meaning of 'if-then', or *the* relation of implication, but rather symbolizes a common partial factor of the various different kinds of implications signified by the 'if-then' phrase.

We can regard the horseshoe as symbolizing a special, extremely weak kind of implication. It is expedient for us to do so, since convenient ways to read '$p \supset q$' are 'if p, then q', 'p implies q', or 'p only if q'. The weak implication symbolized by '$\supset$' is called a *material implication*. Its special name indicates that it is a special concept and not to be confused with the other, more usual kinds of implication. Some conditional statements in English do express merely material implications, as for example 'If Russia is a democracy, then I'm a Dutchman'. It is clear that the implication expressed here is neither logical, definitional, nor causal. No 'real connection' is alleged to hold between what the antecedent states and what is stated by the consequent. This sort of conditional is ordinarily intended as an emphatic or humorous method of denying the truth of its antecedent, for it typically contains a notoriously or ridiculously false statement as its consequent. Any such assertion about truth values is adequately symbolized by using the truth-functional connective '$\supset$'.

Although most conditional statements express more than a merely material implication between antecedent and consequent, we now propose to symbolize *any* occurrence of 'if-then' by the truth-functional connective '$\supset$'. Such symbolizing abstracts from or ignores part of the meaning of most conditional statements. But the proposal can be justified on the grounds that the validity of valid arguments involving conditionals is preserved when the conditionals are regarded as expressing material implications only, as will be established in the following section.

Conditional statements can be expressed in a variety of ways. A statement of the form 'if p, then q' could equally well be expressed as 'if p, q', 'q if p', 'that p implies that q', 'that p entails that q' 'p only if q', 'that p is a sufficient condition that q', or as 'that q is a necessary condition that p', and any of these formulations will be symbolized as $p \supset q$.

EXERCISES

I. If A and B are true statements and X and Y are false statements, which of the following compound statements are true?

*1. $X \supset (X \supset Y)$
2. $(X \supset X) \supset Y$
3. $(A \supset X) \supset Y$
4. $(X \supset A) \supset Y$
*5. $A \supset (B \supset Y)$
6. $A \supset (X \supset B)$
7. $(X \supset A) \supset (B \supset Y)$
8. $(A \supset X) \supset (Y \supset B)$
9. $(A \supset B) \supset (\sim A \supset \sim B)$
*10. $(X \supset Y) \supset (\sim X \supset \sim Y)$
11. $(X \supset A) \supset (\sim X \supset \sim A)$
12. $(X \supset \sim Y) \supset (\sim X \supset Y)$
13. $[(A \cdot X) \supset Y] \supset (A \supset Y)$
14. $[(A \cdot B) \supset X] \supset [A \supset (B \supset X)]$
*15. $[(X \cdot Y) \supset A] \supset [X \supset (Y \supset A)]$
16. $[(A \cdot X) \supset B] \supset [A \supset (B \supset X)]$
17. $[X \supset (A \supset Y)] \supset [(X \supset A) \supset Y]$
18. $[X \supset (X \supset Y)] \supset [(X \supset X) \supset X]$
19. $[(A \supset B) \supset A] \supset A$
20. $[(X \supset Y) \supset X] \supset X$

II. If A and B are known to be true, and X and Y are known to be false, but the truth values of P and Q are not known, of which of the following statements can you determine the truth values?

*1. $X \supset (\sim Y \supset Q)$
2. $\sim Y \supset (P \supset \sim X)$
3. $(P \supset B) \supset Y$
4. $(P \vee A) \supset (Q \cdot X)$
*5. $(P \supset A) \supset (B \supset Y)$
6. $(B \supset P) \supset (Q \supset A)$
7. $A \supset (P \supset A)$
8. $(X \supset Q) \supset X$
9. $\sim(P \cdot X) \supset Y$
*10. $(Q \vee B) \supset X$
11. $[(P \supset A) \supset P] \supset P$
12. $[(P \supset X) \supset P] \supset P$
13. $[(P \supset Q) \supset P] \supset P$
14. $(P \supset X) \supset (X \supset P)$
*15. $(Q \supset B) \supset (A \supset Y)$
16. $(P \supset Q) \supset (\sim Q \supset \sim P)$
17. $(P \supset Q) \supset \{[P \supset (Q \supset A)] \supset (P \supset A)\}$
18. $(P \supset Q) \supset \{[P \supset (Q \supset X)] \supset (P \supset X)\}$
19. $(A \supset P) \supset \{[A \supset (P \supset Q)] \supset (A \supset Q)\}$
20. $(P \supset X) \supset \{[P \supset (X \supset Q)] \supset (P \supset Q)\}$

III. Symbolizing 'Amherst wins its first game' as A, 'Colgate wins its first game' as C, and 'Dartmouth wins its first game' as D, symbolize the following compound statements:

*1. Both Amherst and Colgate win their first games only if Dartmouth does not win its first game.
2. Amherst wins its first game if either Colgate wins its first game or Dartmouth wins its first game.
3. If Amherst wins its first game, then both Colgate and Dartmouth win their first games.
4. If Amherst wins its first game, then either Colgate or Dartmouth wins its first game.
*5. If Amherst does not win its first game, then it is not the case that either Colgate or Dartmouth wins its first game.
6. If it is not the case that both Amherst and Colgate win their first game, then both Colgate and Dartmouth win their first games.
7. If Amherst wins its first game, then not both Colgate and Dartmouth win their first games.
8. If Amherst does not win its first game, then both Colgate and Dartmouth do not win their first games.
9. Either Amherst wins its first game and Colgate does not win its first game or, if Colgate wins its first game, then Dartmouth does not win its first game.
*10. If Amherst wins its first game, then Colgate does not win its first game, but if Colgate does not win its first game, then Dartmouth wins its first game.
11. If Amherst wins its first game, then, if Colgate does not win its first game, then Dartmouth wins its first game.
12. Either Amherst and Colgate win their first games or it is not the case that if Colgate wins its first game, then Dartmouth wins it first game.
13. Amherst wins its first game only if either Colgate or Dartmouth does not win its first game.
14. If Amherst wins its first game only if Colgate wins its first game, then Dartmouth does not win it its first game.
15. If Amherst and Colgate both do not win their first games, then Amherst and Colgate do not both win their first games.

2.3 Argument Forms and Truth Tables

In this section, we develop a purely mechanical method for testing the validity of arguments containing truth-functionally compound statements. That method is closely related to the familiar technique of *refutation by logical analogy* that was used in the first chapter to show the invalidity of the argument

> If I am President, then I am famous.
> I am not President.
> Therefore, I am not famous.

That argument was shown to be invalid by constructing another argument of the same form:

If Rockefeller is President, then Rockefeller is famous.
Rockefeller is not President.
Therefore, Rockefeller is not famous.

which is obviously invalid, since its premisses are true but its conclusion false. Any argument is proved to be invalid if another argument *of exactly the same form* can be constructed with true premisses and a false conclusion. This reflects the fact that validity and invalidity are purely *formal* characteristics of arguments: any two arguments having the same form are either both valid or both invalid, regardless of any differences in their subject matter.[4] The concept of two arguments having *exactly the same form* is one that deserves further examination.

It is convenient, in discussing forms of arguments, to use small letters from the middle part of the alphabet, 'p', 'q', 'r', 's', . . . as *statement variables*, which are defined simply to be letters for which, or in place of which, statements may be substituted. Now, we define an *argument form* to be any array of symbols that contains statement variables, such that when statements are substituted for the statement variables—the same statement being substituted for every occurrence of the same statement variable throughout—the result is an argument. For definiteness, we establish the convention that in any argument form, 'p' shall be the first statement variable that occurs in it, 'q' shall be the second, 'r' the third, and so on.

Any argument which results from the substitution of statements for the statement variables of an argument form is said to *have* that form, or to be a *substitution instance* of that argument form. If we symbolize the simple statement 'The United Nations will become more responsible' as U, and the simple statement 'There will be a third world war' as W, then the Disjunctive Syllogism presented earlier can be symbolized as

$$\begin{array}{l} U \vee W \\ \sim U \\ \therefore W \end{array} \tag{1}$$

It has the form

$$\begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array} \tag{2}$$

from which it results by substituting the statements U and W for the statement variables p and q, respectively. But that is not the only form of which it is a substitution instance. The same argument is obtained by substituting the statements $U \vee W$, $\sim U$, and W for the statement variables p, q, and r, respectively, in the argument form

$$\begin{array}{l} p \\ q \\ \therefore r \end{array} \tag{3}$$

We define *the specific form* of a given argument as that argument form from which the argument results by substituting a different *simple* statement for each distinct statement variable. Thus the specific form of the argument (1) is the argument form (2). Although the argument form (3) is *a* form of the argument (1), it is not *the specific form* of it. The technique of refutation by logical analogy can now be described more precisely. If the specific form of a given argument can be shown to have any substitution instance with true premisses and false conclusion, then the given argument is invalid.

The terms 'valid' and 'invalid' can be extended to apply to argument forms as well as arguments. An *invalid* argument form is one that has at least one substitution instance with true premisses and a false conclusion. The technique of refutation by logical analogy presupposes that any argument of which the specific form is an invalid argument form, is an invalid argument. Any argument form is *valid* that is not invalid: So a *valid* argument form is one that has *no* substitution instance with true premisses and false conclusion. Any given argument can be proved valid if it can be shown that the specific form of the given argument is a valid argument form.

To determine the validity or invalidity of an argument form, we must examine all possible substitution instances of it to see if any of them have true premisses and false conclusions. The arguments with which we are here concerned contain only simple statements and truth-functional compounds of them. We are interested only in the truth values of their premisses and conclusions. We can obtain all possible substitution instances whose premisses and conclusions have different truth values, by considering all possible arrangements of truth values for the statements that are substituted for the distinct statement variables in the argument form to be tested. Possible substitution instances can be set forth most conveniently in a truth table, with an initial or guide column for each distinct statement variable that appears in the argument form. Thus to prove the validity of the Disjunctive Syllogism form

$$\begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array}$$

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[4] Here we assume that the simple statements involved are neither logically true (e.g., 'All equilateral triangles are triangles') nor logically false (e.g., 'Some triangles are nontriangles'). We assume also that the only logical relations among the simple statements involved are those asserted or entailed by the premisses. The point of these restrictions is to limit our considerations in Chapters 2 and 3 to truth-functional arguments alone, and to exclude other kinds of arguments whose validity turns on more complex logical considerations to be introduced in Chapters 4 and 5.

we construct the following truth table:

p	q	$p \vee q$	$\sim p$
T	**T**	**T**	**F**
T	**F**	**T**	**F**
F	**T**	**T**	**T**
F	**F**	**F**	**T**

Each row of this table represents a whole class of substitution instances. The **T**'s and **F**'s in the two initial columns represent the truth values of statements that can be substituted for the variables p and q in the argument form. These determine the truth values in the other columns, the third of which is headed by the first 'premiss' of the argument form, and the fourth by the second 'premiss'. The second column's heading is the 'conclusion' of the argument form. An examination of this truth table reveals that whatever statements are substituted for the variables p and q, the resulting argument cannot have true premisses and a false conclusion. This is so because the third row represents the only possible case in which both premisses are true, and there the conclusion is true also.

Because truth tables provide a purely mechanical or *effective* method of deciding the validity or invalidity of any argument of the general type here considered, we can now justify our proposal to symbolize all conditional statements by means of the truth-functional connective '$\supset$'. The justification for treating all implications as though they were mere material implications is that valid arguments containing conditional statements remain valid when those conditionals are interpreted as expressing material implications only. The three simplest and most intuitively valid forms of argument involving conditional statements are

Modus Ponens If p, then q
p
$\therefore q$

Modus Tollens If p, then q
$\sim q$
$\therefore \sim p$

and the

Hypothetical Syllogism If p, then q
If q, then r
$\therefore$ If p, then r

That they all remain valid when their conditionals are interpreted as expressing material implications is easily established by truth tables. The validity of

Modus Ponens is shown by the same truth table that defines the horseshoe symbol:

p	q	$p \supset q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Here the two premisses are represented by the third and first columns, and the conclusion, by the second. Only the first row represents substitution instances in which both premisses are true, and in that row the conclusion is true also. The validity of *Modus Tollens* is shown by the truth table:

p	q	$p \supset q$	$\sim q$	$\sim p$
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

Here only the fourth row represents substitution instances in which both premisses (the third and fourth columns) are true, and there the conclusion (the fifth column) is true also. Since the Hypothetical Syllogism form contains three distinct statement variables, the truth table for it must have three initial columns and will require eight rows for listing all possible substitution instances:

p	q	r	$p \supset q$	$q \supset r$	$p \supset r$
T	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**F**
T	**F**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**	**T**

In constructing it, the three initial columns represent all possible arrangements of truth values for the statements substituted for the statement variables p, q, and r. The fourth column is filled in by reference to the first and second, the fifth by reference to the second and third, and the sixth by reference to the first and third. The premisses are both true only in the first, fifth, seventh, and eighth rows, and in these rows the conclusion is true also. This suffices to demonstrate that the Hypothetical Syllogism remains valid when its conditionals are symbolized by means of the horseshoe symbol. Any doubts that remain about the claim that valid arguments containing conditionals remain valid when their conditionals are interpreted as expressing merely material implication, can be allayed by the reader's providing, symbolizing, and testing his own examples by means of truth tables.

To test the validity of an argument form by a truth table requires a table with a separate initial or guide column for each different statement variable, and a separate row for every possible assignment of truth values to the statement variables involved. Hence, testing an argument form containing n distinct statement variables, requires a truth table having 2^n rows. In constructing truth tables, it is convenient to fix upon some uniform pattern for inscribing the **T**'s and **F**'s in their initial or guide columns. In this book we shall follow the practice of simply alternating **T**'s and **F**'s down the extreme right-hand initial column and alternating pairs of **T**'s with pairs of **F**'s down the column directly to its left. Next, we shall alternate quadruples of **T**'s with quadruples of **F**'s, . . . , and finally, we shall fill in the top half of the extreme left-hand initial column with **T**'s and its bottom half with **F**'s.

There are two invalid argument forms that bear a superficial resemblance to the valid argument forms *Modus Ponens* and *Modus Tollens*. These are

$$\begin{array}{lcl} p \supset q & & p \supset q \\ q & \text{and} & \sim p \\ \therefore p & & \therefore \sim q \end{array}$$

and are known as the Fallacies of Affirming the Consequent and of Denying the Antecedent, respectively. The invalidity of both can be shown by a single truth table:

p	q	$p \supset q$	$\sim p$	$\sim q$
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	**F**
F	**F**	**T**	**T**	**T**

The two premisses in the Fallacy of Affirming the Consequent head the second and third columns, and are true in both the first and third rows. But the conclusion, which heads the first column, is false in the third row—which shows that the argument form does have a substitution instance with true premisses and a false conclusion, and is therefore invalid. Columns three and four are headed by the two premisses in the Fallacy of Denying the Antecedent, which are true in both the third and fourth rows. Its conclusion heads the fifth column and is false in the third row—which shows that the second argument form is invalid also.

It must be emphasized that although a valid argument form has only valid arguments as substitution instances, an invalid argument form can have both

valid and invalid substitution instances. So to prove that a given argument is invalid, we must prove that *the specific form* of that argument is invalid.

EXERCISES

I. For each of the following arguments indicate which, if any, of the argument forms in Exercise II below have the given argument as a substitution instance, and indicate which, if any, is the specific form of the given argument:

***a.** A
$\therefore A \vee B$

b. $C \cdot D$
$\therefore C$

c. $E \supset (F \cdot G)$
$\therefore \sim(F \cdot G) \supset \sim E$

d. H
I
$\therefore H \cdot I$

***e.** $J \supset (K \cdot L)$
$J \vee (K \cdot L)$
$\therefore K \cdot L$

f. $M \supset (N \supset O)$
$O \supset \sim M$
$\therefore O \supset \sim N$

g. $(P \supset Q) \cdot (R \supset S)$
$\therefore P \supset Q$

h. $T \supset U$
$\therefore (T \supset U) \vee (V \cdot T)$

i. $W \supset X$
$\therefore X \supset (W \supset X)$

***j.** $Y \vee (Z \cdot \sim Y)$
Y
$\therefore \sim(Z \cdot \sim Y)$

k. $(A \supset B) \cdot (C \supset D)$
$A \vee C$
$\therefore B \vee D$

l. $(E \supset F) \cdot (G \supset H)$
$\sim F \vee \sim G$
$\therefore \sim E \vee \sim H$

m. $I \supset J$
$\therefore (I \supset J) \supset (I \supset J)$

n. $K \supset (L \supset M)$
$K \supset L$
$\therefore K \supset M$

o. $N \supset (N \supset O)$
$N \supset N$
$\therefore N \supset O$

II. Use truth tables to determine the validity or invalidity of each of the following argument forms:

***1.** $p \cdot q$
$\therefore p$

2. p
$\therefore p \cdot q$

3. $p \vee q$
$\therefore p$

4. p
$\therefore p \vee q$

***5.** p
$\therefore p \supset q$

6. p
$\therefore q \supset p$

7. $p \supset q$
$\therefore \sim q \supset \sim p$

8. $p \supset q$
$\therefore \sim p \supset \sim q$

9. $p \supset (q \cdot r)$
$\therefore \sim(q \cdot r) \supset \sim p$

***10.** $p \vee q$
p
$\therefore \sim q$

11. p
q
$\therefore p \cdot q$

12. $p \supset q$
$q \supset p$
$\therefore p \vee q$

13. $p \supset q$
$p \vee q$
$\therefore q$

14. $p \supset (q \supset r)$
$p \supset q$
$\therefore p \supset r$

***15.** $(p \supset q) \cdot (p \supset r)$
p
$\therefore q \vee r$

16. $p \supset (q \vee r)$
$p \supset \sim q$
$\therefore p \vee r$

17. $(p \supset q) \cdot (r \supset s)$
$p \vee r$
$\therefore q \vee s$

18. $(p \supset q) \cdot (r \supset s)$
$\sim q \vee \sim s$
$\therefore \sim p \vee \sim r$

19. $(p \vee q) \supset (p \cdot q)$
$p \cdot q$
$\therefore p \vee q$

20. $p \vee (q \cdot \sim p)$
p
$\therefore \sim(q \cdot \sim p)$

21. $(p \vee q) \supset (p \cdot q)$
$\sim(p \vee q)$
$\therefore \sim(p \cdot q)$

III. Use truth tables to determine the validity or invalidity of each of the following arguments:

*1. If Alice is elected class president, then either Betty is elected vice-president or Carol is elected treasurer. Betty is elected vice-president. Therefore, if Alice is elected class-president, then Carol is not elected treasurer.

2. If Alice is elected class president, then either Betty is elected vice-president or Carol is elected treasurer. Carol is not elected treasurer. Therefore, if Betty is not elected vice-president, then Alice is not elected class president.

3. If Alice is elected class president, then Betty is elected vice-president and Carol is elected treasurer. Betty is not elected vice-president. Therefore, Alice is not elected class president.

4. If Alice is elected class president, then if Betty is elected vice-president, then Carol is elected treasurer. Betty is not elected vice-president. Therefore, either Alice is elected class president or Carol is elected treasurer.

*5. If the seed catalog is correct, then if the seeds are planted in April, then the flowers bloom in July. The flowers do not bloom in July. Therefore, if the seeds are planted in April, then the seed catalog is not correct.

6. If the seed catalog is correct, then if the seeds are planted in April, then the flowers bloom in July. The flowers bloom in July. Therefore, if the seed catalog is correct, then the seeds are planted in April.

7. If the seed catalog is correct, then if the seeds are planted in April, then the flowers bloom in July. The seeds are planted in April. Therefore, if the flowers do not bloom in July, then the seed catalog is not correct.

8. If the seed catalog is correct, then if the seeds are planted in April, then the flowers bloom in July. The flowers do not bloom in July. Therefore, if the seeds are not planted in April, then the seed catalog is not correct.

9. If Ed wins first prize, then Fred wins second prize, and if Fred wins second prize, then George is disappointed. Either Ed wins first prize or George is disappointed. Therefore, Fred does not win second prize.

*10. If Ed wins first prize, then either Fred wins second prize or George is disappointed. Fred does not win second prize. Therefore, if George is disappointed, then Ed does not win first prize.

11. If Ed wins first prize, then Fred wins second prize, and if Fred wins second prize, then George is disappointed. Either Fred does not win second prize or George is not disappointed. Therefore, Ed does not win first prize.

12. If Ed wins first prize, then Fred wins second prize, and if Fred wins second prize, then George is disappointed. Either Ed does not win first prize or Fred does not win second prize. Therefore, either Fred does not win second prize or George is not disappointed.

13. If the weather is warm and the sky is clear, then we go swimming and we go boating. It is not the case that if the sky is clear, then we go swimming. Therefore, the weather is not warm.

14. If the weather is warm and the sky is clear, then either we go swimming or we go boating. It is not the case that if the sky is clear, then we go swimming. Therefore, if we do not go boating, then the weather is not warm.

*15. If the weather is warm and the sky is clear, then either we go swimming or we go boating. It is not the case that if we do not go swimming, then the sky is not clear. Therefore, either the weather is warm or we go boating.

2.4 Statement Forms

The introduction of statement variables in the preceding section enabled us to define both argument forms in general and the specific form of a given argument. Now, we define a *statement form* to be any sequence of symbols containing statement variables, such that when statements are substituted for the statement variables—the same statement being substituted for every occurrence of the same statement variable throughout—the result is a statement. Again for definiteness, we establish the convention that in any statement form, '*p*' shall be the first statement variable that occurs in it. The statement variable '*q*' shall be the second, '*r*' the third, and so on. Any statement which results from substituting statements for the statement variables of a statement form is said to *have* that form, or to be a *substitution instance* of it. Just as we distinguished the specific form of a given argument, so we distinguish *the specific form* of a given statement as that statement form from which the given statement results by substituting a different simple statement for each distinct statement variable. For example, where *A*, *B*, and *C* are different simple statements, the compound statement $A \supset (B \vee C)$ is a substitution instance of the statement form $p \supset q$, and also of the statement form $p \supset (q \vee r)$, but only the latter is the specific form of the given statement.

Although the statements 'Balboa discovered the Pacific Ocean' (*B*) and 'Balboa discovered the Pacific Ocean or else he didn't' ($B \vee \sim B$) are both true, we discover their truth in quite different ways. The truth of *B* is a matter of history, and must be learned through empirical investigation. Moreover, events might possibly have been such as to make *B* false; there is nothing *necessary* about the truth of *B*. But the truth of the statement $B \vee \sim B$ *can* be known independently of empirical investigation, and no events could possibly have made it false, for it is a necessary truth. The statement $B \vee \sim B$ is a formal truth, a substitution instance of a statement form *all* of whose substitution instances are true. A statement form that has only true substitution instances is said to be *tautologous*, or a *tautology*. The specific form of $B \vee \sim B$ is $p \vee \sim p$, and is proved a tautology by the following truth table:

p	$\sim p$	$p \vee \sim p$
T	**F**	**T**
F	**T**	**T**

That there are only **T**'s in the column headed by the statement form in question shows that all of its substitution instances are true. Any statement that is a substitution instance of a tautologous statement form is formally true and is, itself, said to be tautologous or a tautology.

Similarly, although the statements 'Cortez discovered the Pacific' (*C*) and 'Cortez discovered the Pacific and Cortez did not discover the Pacific' ($C \cdot \sim C$) are both false, we discover their falsehood in quite different ways. The first simply *happens* to be false, and that must be learned empirically; the second is necessarily false, and that can be known independently of empirical investigation. The statement $C \cdot \sim C$ is formally false, a substitution instance of a statement form, *all* of whose substitution instances are false. One statement is said to contradict, or to be a contradiction of, another statement when it is logically impossible for them both to be true. In this sense, *contradiction* is a relation between statements. But there is another, related sense of that term. When it is logically impossible for a particular statement to be true, that statement itself is said to be self-contradictory, or a self-contradiction. Such statements are also said more simply to be contradictory or contradictions. We shall follow the latter usage here. A statement form that has only false substitution instances is said to be *contradictory* or a *contradiction*, and the same terms are applied to its substitution instances. The statement form $p \cdot \sim p$ is proved a contradiction by the fact that in its truth table only **F**'s occur in the column that it heads.

Statements and statement forms that are neither tautologous nor contradictory are said to be *contingent* or *contingencies*. For example, p, $\sim p$, $p \vee q$, $p \cdot q$, and $p \supset q$ are contingent statement forms; and B, C, $\sim B$, $\sim C$, $B \cdot C$, $B \vee C$ are contingent *statements*. The term is appropriate, since their truth values are not formally determined but are dependent or contingent upon what happens to be the case.

It is easily proved that $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$ are tautologies. When expressed in English as 'A true statement is implied by any statement whatever', and as 'A false statement implies any statement whatever', they seem rather strange. They have been called by some writers the *paradoxes of material implication*. When it is kept in mind that the horseshoe symbol is a truth-functional connective that stands for *material* implication rather than either 'implication in general' or more usual kinds of implications such as logical or causal, then the tautologous statement forms in question are not at all surprising. And when the misleading English formulations are corrected by inserting the word 'materially' before 'implied' and 'implies', then the air of paradox vanishes. Material implication is a special, technical concept that the logician introduces and uses because it simplifies his task of discriminating valid from invalid arguments.

Two statements are said to be *materially equivalent* when they have the same truth value, and we symbolize the statement *that they are* materially equivalent by inserting the symbol '$\equiv$' between them. Being a truth-functional connective, the three-bar symbol is defined by the following truth table:

p	q	$p \equiv q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

To say that two statements are materially equivalent, is to say that they materially imply each other, as is easily verified by a truth table. Hence the three-bar symbol may be read 'if and only if'. A statement or statement form of the pattern $p \equiv q$ is called a *biconditional*. Two statements are said to be *logically equivalent* when the biconditional that expresses their material equivalence is a tautology. The 'principle of Double Negation', expressed as $p \equiv \sim\sim p$, is proved to be tautologous by a truth table.

There are two logical equivalences that express important interrelations of conjunctions, disjunctions, and negations. Because a conjunction asserts that both its conjuncts are true, its negation need assert only that at least one is false. Thus negating the conjunction $p \cdot q$ amounts to asserting the disjunction of the negations of p and q. This statement of equivalence is symbolized as $\sim(p \cdot q) \equiv (\sim p \vee \sim q)$, and proved to be a tautology by the following truth table:

p	q	$p \cdot q$	$\sim(p \cdot q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \cdot q) \equiv (\sim p \vee \sim q)$
T	**T**	**T**	**F**	**F**	**F**	**F**	**T**
T	**F**	**F**	**T**	**F**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**F**	**T**	**T**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**

Similarly, because a disjunction asserts merely that at least one disjunct is true, to negate it is to assert that both are false. Negating the disjunction $p \vee q$ amounts to asserting the conjunction of the negations of p and q. It is symbolized as $\sim(p \vee q) \equiv (\sim p \cdot \sim q)$ and is easily proved tautologous by a truth table. These two equivalences are known as De Morgan's Theorems, after the English mathematician-logician Augustus De Morgan (1806–1871) and can be stated compendiously in English as:

> The negation of the $\left\{\begin{matrix}\text{conjunction}\\ \text{disjunction}\end{matrix}\right\}$ of two statements is logically equivalent to the $\left\{\begin{matrix}\text{disjunction}\\ \text{conjunction}\end{matrix}\right\}$ of their negations.

Two statement forms are said to be *logically* equivalent if no matter what statements are substituted for their statement variables—the same statement being substituted for the same statement variable in both statement forms—the resulting pairs of statements are materially equivalent. Thus De Morgan's Theorem asserts that $\sim(p \vee q)$ and $\sim p \cdot \sim q$ are logically equivalent statement forms. By De Morgan's Theorem and the principle of Double Negation $\sim(p \cdot \sim q)$ and $\sim p \vee q$ are logically equivalent, hence either can be taken as defining $p \supset q$. The second alternative is the more usual choice.

To every argument corresponds a conditional statement whose antecedent is the conjunction of the argument's premisses and whose consequent is the argument's conclusion. That corresponding conditional is a tautology if and only if the argument is valid. Thus to the valid argument form

$p \vee q$
$\sim p$
$\therefore q$

corresponds the tautologous statement form $[(p \vee q) \cdot \sim p] \supset q$; and to the invalid argument form

$p \supset q$
q
$\therefore p$

corresponds the nontautologous statement form $[(p \supset q) \cdot q] \supset p$. An argument form is valid if and only if its truth table has a **T** under its conclusion in every row in which there are **T**'s under all of its premisses. Since an **F** can occur in the column headed by its corresponding conditional statement only where there are **T**'s under all of those premisses and an **F** under the conclusion, it is clear that there can be only **T**'s under a conditional that corresponds to a valid argument form. If an argument is valid, the statement that the conjunction of its premisses implies its conclusion is a tautology.

An alternative version of the truth table test of a statement form is the following, which corresponds to the preceding truth table.

$\sim$	$(p$	$\cdot$	$q)$	$\equiv$	$(\sim$	p	$\vee$	$\sim$	$q)$
F	**T**	**T**	**T**	**T**	**F**	**T**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**T**	**F**	**T**	**F**	**T**
T	**F**	**F**	**F**	**T**	**T**	**F**	**T**	**T**	**F**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)

Here columns (2), (4), (7), (10) are the initial or guide columns. Column (3) is filled in by reference to columns (2) and (4), and column (1) by reference to column (3). Column (6) is filled in by reference to column (7), column (9) is filled in by reference to column (10), and then column (8) by reference to columns (6) and (9). Finally, column (5) is filled in by reference to columns (1) and (8). That its main connective has only **T**'s under it in the truth table establishes that the statement form being tested is a tautology.

EXERCISES

I. Use truth tables to characterize the following statement forms as tautologous, contradictory, or contingent:

***1.** $p \supset \sim p$
2. $(p \supset \sim p) \cdot (\sim p \supset p)$
3. $p \supset (p \supset p)$
4. $(p \supset p) \supset p$

***5.** $p \supset (p \cdot p)$
6. $(p \cdot q) \supset p$
7. $(p \supset q) \supset [\sim(q \cdot r) \supset \sim(r \cdot p)]$
8. $(\sim p \cdot q) \cdot (q \supset p)$
9. $[(p \supset q) \supset q] \supset q$
10. $[(p \supset q) \supset p] \supset p$

II. Use truth tables to decide which of the following are logical equivalences:

***1.** $(p \supset q) \equiv (\sim p \supset \sim q)$
2. $(p \supset q) \equiv (\sim q \supset \sim p)$
3. $[(p \cdot q) \supset r] \equiv [p \supset (q \supset r)]$
4. $[p \supset (q \supset r)] \equiv [(p \supset q) \supset r]$
***5.** $[p \cdot (q \vee r)] \equiv [(p \cdot q) \vee (p \cdot r)]$
6. $[p \vee (q \cdot r)] \equiv [(p \vee q) \cdot r]$
7. $[p \vee (q \cdot r)] \equiv [(p \vee q) \cdot (p \vee r)]$
8. $(p \equiv q) \equiv [(p \cdot q) \vee (\sim p \cdot \sim q)]$
9. $p \equiv [p \cdot (p \supset q)]$
10. $p \equiv [p \cdot (q \supset p)]$