
Second Year Algebra

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Contents

1

Quadratic Equations and Functions

1

1.1	Simple quadratic equations	2
1.2	Solving quadratic equations by factoring	3
1.3	Completing the square	4
1.4	Quadratic formula	7
1.5	Further uses of the quadratic formula	8
1.6	Solving general problems involving quadratic equations	9
1.7	Finding quadratic equations	11
1.8	Graphs of quadratic functions	13
1.9	Graphs of quadratic functions, continued	15
1.10	Applications of quadratic functions	16
1.11	Geometric definition of a parabola	18
1.12	Reflection property of parabolas	21
1.13	Quadratic inequalities	23

1

Quadratic Equations and Functions

As a quick reminder, a linear equation is an equation that can be put in the form $ax + b = 0$, where a and b are constants with $a \neq 0$. It is easy to see that $x = -b/a$ is the one and only solution to this equation. A linear function is a function of the form $f(x) = ax + b$ (with $a \neq 0$) and, as we have seen, the graph of a linear function is a straight line. The solution to the linear equation $ax + b = 0$ represents the x -coordinate of the point on the graph of $y = ax + b$ at which the line crosses the x -axis, that is, the solution to the equation gives the x -intercept of the corresponding line.

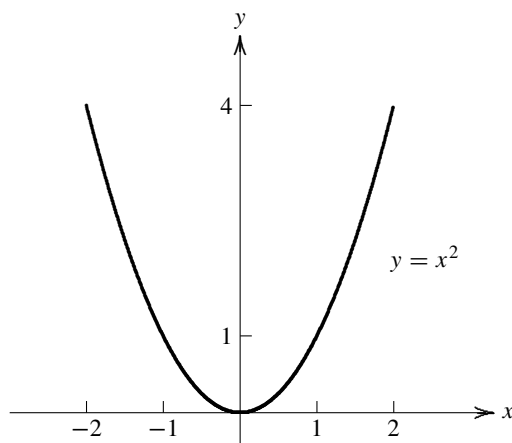
A quadratic equation is an equation that can be put in the form

$$ax^2 + bx + c = 0,$$

where a , b , and c are constants with $a \neq 0$. More work is required to solve equations of this type and in this chapter we will look at several different ways to solve quadratic equations. Unlike linear equations which always have one solution, we will see that a quadratic equation may have two, one, or no solutions. A quadratic function is a function of the form $f(x) = ax^2 + bx + c$ (with $a \neq 0$). The graph of a general function such as $y = ax^2 + bx + c$ is similar to the graph of the simplest quadratic function, namely, $y = x^2$. Plotting a few points reveals a curve known as a parabola (see the graph on the next page). We will study the features of this graph and show how the graph of a general quadratic function is related to the graph of the simple function $f(x) = x^2$. As with linear functions and equations, the solution to a quadratic equation gives the x -intercepts of the graph of the corresponding function. By translating the graph of a generic parabola, you should be able to see how a

2 Chapter 1 Quadratic Equations and Functions

quadratic equation can have two, one, or no solutions.



Parabolic curves have some interesting properties. To mention two examples, (1) a ball thrown into the air follows a path that is in the shape of a parabola and (2) telescope lenses and car headlights are often designed with parabolic cross-sections because of the nature of the reflective properties of the parabola.

1.1 SIMPLE QUADRATIC EQUATIONS

Given a positive number a , the symbol \sqrt{a} represents the square root of a . This means that

$$\sqrt{a} \cdot \sqrt{a} = a \quad \text{or} \quad (\sqrt{a})^2 = a.$$

For example, it is easy to see that $\sqrt{9} = 3$, $\sqrt{121} = 11$, and $\sqrt{49/16} = 7/4$. However, it is not possible to write down the square root of most numbers in an exact form. For instance,

$$\sqrt{2} \approx 1.41421356 \quad \text{but} \quad 1.41421356^2 = 1.999999932878736 \neq 2.$$

This is due to the fact that $\sqrt{2}$ is an irrational number; it cannot be represented as a ratio of two integers and its decimal representation does not terminate or have a repeating pattern. For this reason, we often leave the square root of 2 in the form $\sqrt{2}$ rather than try to write out part of its decimal representation. It is important to remember that symbols such as $\sqrt{2}$, $\sqrt{73}$, and $\sqrt{6.031}$ represent numbers with a certain property even if we do not write them out as decimals.

Suppose we need to find a number x with the property that $x^2 = 16$. One solution is 4, but another correct solution is -4 , that is, both $4 \cdot 4 = 16$ and $(-4) \cdot (-4) = 16$. Which of these numbers is the square root of 16? By convention, the symbol $\sqrt{16}$ represents the positive square root of 16. This means that the correct solution to the equation $x^2 = 16$ is $x = \pm\sqrt{16} = \pm 4$; the \pm sign is needed to include both solutions.

Using these ideas, we can solve equations such as $(x - 8)^2 = 36$:

$$(x - 8)^2 = 36 \quad \Rightarrow \quad x - 8 = \pm\sqrt{36} \quad \Rightarrow \quad x - 8 = \pm 6 \quad \Rightarrow \quad x = 8 \pm 6$$

The solutions are thus $x = 2$ and $x = 14$. To solve $3(x - 2)^2 - 96 = 0$, we would do the following steps:

$$\begin{aligned} 3(x - 2)^2 - 96 = 0 &\Rightarrow 3(x - 2)^2 = 96 \Rightarrow (x - 2)^2 = 32 \\ &\Rightarrow x - 2 = \pm\sqrt{32} = \pm 4\sqrt{2} \Rightarrow x = 2 \pm 4\sqrt{2} \end{aligned}$$

Note that it is important to get the squared term by itself before taking square roots. The solutions in this case are thus $x = 2 + 4\sqrt{2}$ and $x = 2 - 4\sqrt{2}$. If decimal approximations are needed for these numbers, we can use a calculator and find that the solutions are $x \approx 7.657$ and $x \approx -3.657$, rounded to the nearest thousandth.

Exercises

1. Solve each of the following equations.

a) $x^2 = 144$

b) $3x^2 = 48$

c) $2x^2 = 250$

d) $5x^2 - 3 = 72$

e) $4x^2 + 11 = 103$

f) $41 - 7x^2 = 6$

g) $(x - 7)^2 = 81$

h) $4(x + 2)^2 = 400$

i) $5(x - 1)^2 = 605$

2. Solve each of the following equations.

a) $4x^2 - 49 = 0$

b) $64 - 81x^2 = 0$

c) $(3x)^2 = 180$

d) $\left(x - \frac{3}{2}\right)^2 = \frac{169}{4}$

e) $(5x - 1)^2 - 121 = 0$

f) $(2x + 3)^2 = 49$

g) $4(3x + 7)^2 = 800$

h) $9(x - 5)^2 = 289$

i) $(x - 2\sqrt{5})^2 = 2000$

3. The length of a rectangle is three times its width. Find the dimensions of the rectangle if its area is 1500 square inches.

4. The area of an equilateral triangle is 300 square feet. Find the length of the side of the triangle.

5. The area of a circle is 40π square meters. Find the circumference of the circle.

6. The area of a circle is 100 square feet. Find the diameter of the circle.

1.2 SOLVING QUADRATIC EQUATIONS BY FACTORING

Some quadratic equations can be solved by factoring. For example, suppose we need to solve the equation $x^2 + x - 20 = 0$. Since

$$x^2 + x - 20 = (x + 5)(x - 4),$$

we need the product $(x + 5)(x - 4)$ to be 0. The only way that this can happen is if $x = -5$ or $x = 4$. These are the solutions we are seeking. A similar approach works for any quadratic expression that can be factored. The key idea is the following theorem.

THEOREM 1.1 Zero Product Property If a and b are real numbers such that $ab = 0$, then either $a = 0$ or $b = 0$ or both a and b are zero. ■

It is important to realize how important the number 0 is in this theorem. Suppose you are given two numbers whose product is 12. Can you figure out what the numbers are? Most student might think of products such as

$$1 \cdot 12 = 12, \quad 2 \cdot 6 = 12, \quad 3 \cdot 4 = 12, \quad (-1) \cdot (-12) = 12, \quad (-2) \cdot (-6) = 12, \quad (-3) \cdot (-4) = 12,$$

4 Chapter 1 Quadratic Equations and Functions

but there are many other possibilities such as

$$\frac{1}{2} \cdot 24 = 12, \quad \frac{5}{3} \cdot \frac{36}{5} = 12, \quad \frac{\pi}{4} \cdot \frac{48}{\pi} = 12, \quad \sqrt{18} \cdot \sqrt{8} = 12.$$

The point here is that there are infinitely many ways to find two numbers whose product is 12. However, there are very few ways to find two numbers whose product is 0; at least one of the numbers must be 0.

Since you have had practice factoring trinomials, we only do one more example. To solve $6x^2 + 7 = 23x$, the first step is to set the equation equal to 0. After that, we proceed by factoring and using the zero product property.

$$6x^2 + 7 = 23x \Rightarrow 6x^2 - 23x + 7 = 0 \Rightarrow (2x - 7)(3x - 1) = 0.$$

It follows that either $2x - 7 = 0$ or $3x - 1 = 0$. The solutions to the equation are $x = 7/2$ and $x = 1/3$.

Exercises

1. Solve each of the following equations.

a) $x^2 - 4x + 3 = 0$

b) $x^2 - 5x - 14 = 0$

c) $x^2 + 3x - 40 = 0$

d) $x^2 - 4x = 32$

e) $x^2 + 11 = 12x$

f) $6x - x^2 = 9$

g) $2x^2 + 5x = 12$

h) $4x^2 + 15 = 17x$

i) $10x^2 = 35 - 11x$

2. Solve each of the following equations.

a) $x(x - 7) = 8$

b) $x(x + 3) = 130$

c) $(x - 2)^2 = 2x^2 + 6x + 20$

d) $x(x + 3) = 35 - x^2$

e) $\frac{4}{x} = \frac{x}{x - 1}$

f) $\frac{x}{x - 14} = \frac{3}{x - 10}$

g) $x^4 - 10x^2 + 9 = 0$

h) $x^4 - 8x^2 - 9 = 0$

i) $\frac{12}{x} + 5 = x + 1$

3. The width of a rectangle is three less than its length. Find the dimensions of the rectangle if its area is 180 square meters.

4. The length of a rectangle is three more than twice its width. Find the dimensions of the rectangle if its area is 230 square feet.

1.3 COMPLETING THE SQUARE

Consider the squares of the following binomials:

$$\begin{aligned}(x + 3)^2 &= x^2 + 6x + 9; \\ \left(x + \frac{11}{2}\right)^2 &= x^2 + 11x + \frac{121}{4}; \\ (x + k)^2 &= x^2 + 2kx + k^2;\end{aligned}$$

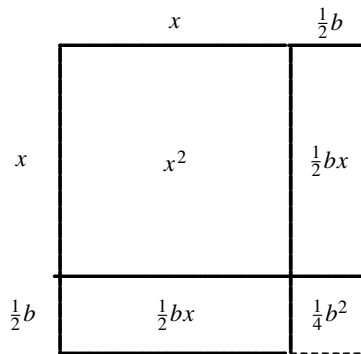
$$\begin{aligned}(x - 7)^2 &= x^2 - 14x + 49; \\ \left(x - \frac{5}{4}\right)^2 &= x^2 - \frac{5}{2}x + \frac{25}{16}; \\ \left(x - \frac{b}{2}\right)^2 &= x^2 - bx + \left(\frac{b}{2}\right)^2.\end{aligned}$$

Notice that in each of the trinomial forms of these squares, the constant term is the square of half of the coefficient of the linear term. With these examples in mind, can you find a value of c that will turn each of the following trinomials into a perfect square?

$$x^2 + 8x + c, \quad x^2 - 10x + c, \quad x^2 - 7x + c, \quad x^2 + \frac{5}{3}x + c, \quad x^2 + bx + c.$$

You should find that the desired values of c are 16, 25, $49/4$, $25/36$, and $b^2/4$, respectively.

Finding an appropriate value for c to turn $x^2 + bx$ into a perfect square is known as completing the square. The following picture reveals a geometric interpretation of the algebra that is being performed. We think of $x^2 + bx$ as the area of a square with side length x along with two rectangles attached to it with length x and width $b/2$. The resulting figure needs a patch in the bottom right corner to turn it into a square; the size of this patch is a square with side $b/2$.



After the region with area $b^2/4$ is filled in, we have a square with a side of $x + (b/2)$. In other words, the area of the whole figure is the sum of the areas of the larger square, the two rectangles, and the little square:

$$\left(x + \frac{b}{2}\right)^2 = x^2 + \frac{1}{2}bx + \frac{1}{2}bx + \left(\frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$$

Therefore, the algebra step of adding an extra term simply fills in the remaining part of the square; this is why we refer to this technique as “completing the square.”

We can use this idea of completing the square to solve quadratic equations. For example, consider the following problem and the steps used to solve it.

$$\begin{array}{ll} x^2 - 12x = 253 & \text{given problem} \\ x^2 - 12x + 36 = 253 + 36 & \text{complete the square, } (-12/2)^2 = 36 \\ (x - 6)^2 = 289 & \text{simplify} \\ x - 6 = \pm 17 & \text{take square roots} \\ x = 6 \pm 17 & \text{solve for } x \end{array}$$

The solutions to the equation are thus $x = 23$ and $x = -11$. Remember to use \pm when you take square roots in an equation. For a second example, consider the following:

$$\begin{array}{ll} x^2 = 135 - 10x & \text{given problem} \\ x^2 + 10x = 135 & \text{get } x \text{ terms on one side} \\ x^2 + 10x + 25 = 135 + 25 & \text{complete the square, } (10/2)^2 = 25 \end{array}$$

$$\begin{aligned}(x + 5)^2 &= 160 && \text{simplify} \\ x + 5 &= \pm 4\sqrt{10} && \text{take square roots} \\ x &= -5 \pm 4\sqrt{10} && \text{solve for } x\end{aligned}$$

The solutions in this case are $-5 + 4\sqrt{10}$ and $-5 - 4\sqrt{10}$.

The technique of completing the square is only valid when the x^2 coefficient is 1. If the coefficient of x^2 is any number other than 1, then the first step is to divide the equation by an appropriate number to obtain a coefficient of 1. This is illustrated in the following example.

$$\begin{aligned}-2x^2 + 9x - 4 &= 0 && \text{given problem} \\ x^2 - \frac{9}{2}x + 2 &= 0 && \text{divide through by } -2 \\ x^2 - \frac{9}{2}x + \frac{81}{16} &= -2 + \frac{81}{16} && \text{complete the square} \\ \left(x - \frac{9}{4}\right)^2 &= \frac{49}{16} && \text{simplify} \\ x - \frac{9}{4} &= \pm \frac{7}{4} && \text{take square roots} \\ x &= \frac{9}{4} \pm \frac{7}{4} && \text{solve for } x\end{aligned}$$

The solutions to this problem are $x = 4$ and $x = 1/2$.

Exercises

1. Find a value for c to turn the given trinomial into a perfect square. Replace c with this value and write the resulting trinomial as a perfect square.

a) $x^2 - 2x + c$

b) $x^2 + 18x + c$

c) $x^2 - 40x + c$

d) $x^2 + 3x + c$

e) $x^2 - \frac{7}{4}x + c$

f) $x^2 + \frac{5}{6}x + c$

g) $x^2 - 4\sqrt{7}x + c$

h) $x^2 + \sqrt{20}x + c$

i) $x^2 + kx + c$

2. Solve each of the following equations by completing the square.

a) $x^2 - 2x - 8 = 0$

b) $x^2 + 4x - 60 = 0$

c) $x^2 - 6x - 112 = 0$

d) $x^2 - 2x - 7 = 0$

e) $x^2 - 4x + 1 = 0$

f) $x^2 - 6x - 20 = 0$

g) $x^2 - 12x = 27$

h) $x^2 + 14x = 50$

i) $x^2 - 16x = 56$

j) $x^2 + \sqrt{20}x = 19$

k) $x^2 + 4\sqrt{7}x - 56 = 0$

l) $x^2 - \sqrt{48}x = 71$

3. Solve each of the following equations by completing the square.

a) $x(x - 8) = 209$

b) $x(20 - x) = 40$

c) $x(x + 12) = 100$

d) $x^2 - 3x + 1 = 0$

e) $x^2 - x = 10$

f) $x^2 + 5x - 12 = 0$

g) $x(x - 7) = 20$

h) $x(15 - x) = 4$

i) $x(x + 9) = 55$

4. Solve each of the following equations by completing the square.

$$\begin{array}{lll} \text{a) } 4x^2 - 24x = 100 & \text{b) } 5x^2 - 40x - 100 = 0 & \text{c) } 17x - 2x^2 = 21 \\ \text{d) } 2x^2 - 5x = 4 & \text{e) } 3x^2 - 6x - 13 = 40 & \text{f) } x(5x - 3) = 10 \end{array}$$

5. The width of a rectangle is six less than its length. Find the dimensions of the rectangle if its area is 121 square meters.

1.4 QUADRATIC FORMULA

As some of the exercises in the last section indicate, completing the square can become tedious when fractions are involved. Since the steps for completing the square are the same for every problem, it is possible to do them with generic coefficients and obtain a formula that will give the solution to the equation. This formula is known as the quadratic formula. The following steps give a derivation of this formula; see if you can understand each of them.

$$\begin{aligned} ax^2 + bx + c = 0 &\Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\ &\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a} \\ &\Leftrightarrow x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\ &\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\ &\Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &\Leftrightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\ &\Leftrightarrow x = -\frac{b}{2a} + \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\ &\Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Thus the solutions to the equation $ax^2 + bx + c = 0$ are given by the last formula; it is referred to as the quadratic formula. It would be a very good idea to memorize this formula.

As a simple example of the use of this formula, consider the equation $x^2 - 5x - 24 = 0$. You should be able to solve this equation by factoring and find that the solutions are $x = -3$ and $x = 8$. To solve this quadratic equation with the quadratic formula, we note that $a = 1$, $b = -5$, and $c = -24$, then compute

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-24)}}{2(1)} = \frac{5 \pm \sqrt{25 + 96}}{2} = \frac{5 \pm 11}{2}.$$

The solutions are thus $x = (5 + 11)/2 = 8$ and $x = (5 - 11)/2 = -3$ as expected.

8 Chapter 1 Quadratic Equations and Functions

As an example of quadratic equation for which the quadratic formula is quite helpful, consider the equation $3x^2 = 52 - 4x$. To solve this quadratic equation with the quadratic formula, we do the following steps.

$$\begin{aligned}3x^2 &= 52 - 4x && \text{given problem} \\3x^2 + 4x - 52 &= 0 && \text{set equal to 0} \\x &= \frac{-4 \pm \sqrt{4^2 - 4(3)(-52)}}{2(3)} && \text{use quadratic formula} \\x &= \frac{-4 \pm \sqrt{16 + 624}}{6} && \text{simplify} \\x &= \frac{-4 \pm 8\sqrt{10}}{6} && \text{simplify} \\x &= \frac{-2 \pm 4\sqrt{10}}{3} && \text{simplify} \\x &= -\frac{2}{3} \pm \frac{4}{3}\sqrt{10} && \text{simplify}\end{aligned}$$

To the nearest hundredth, the solutions are $x \approx 3.55$ and $x \approx -4.88$.

Referring to the quadratic formula, the term $b^2 - 4ac$ is called the discriminant. When the discriminant is positive, the quadratic equation has two distinct real solutions. When the discriminant is 0, the quadratic equation has one real solution; we typically think of this as two solutions that just happen to have the same value. Finally, if the discriminant is negative, then the quadratic equation has no real solutions.

Exercises

1. Solve each of the following equations. Give both the exact solutions and the solutions to the nearest hundredth.

- | | | |
|---------------------------|-------------------------------|---------------------------|
| a) $x^2 - 2x - 17 = 0$ | b) $2x^2 - 5x - 20 = 0$ | c) $x^2 - 6x - 20 = 0$ |
| d) $3x^2 - 4x = 27$ | e) $x^2 - 7x = 20$ | f) $5x^2 - 6x = 80$ |
| g) $2x(x - 8) = 75$ | h) $3x(20 - x) = 40$ | i) $x(2x - 15) = 100$ |
| j) $2x(x - 4) = 5(2 - x)$ | k) $x(20 - x) = x^2 + 4x - 1$ | l) $x(2x - 5) = 4(x + 3)$ |

2. Use the discriminant to determine the number of distinct real solutions for each equation.

- | | | |
|------------------------|--------------------|-----------------------|
| a) $3x^2 - 4x - 7 = 0$ | b) $4x^2 - x = 15$ | c) $x(x + 12) = -80$ |
| d) $9x^2 + 25 = 30x$ | e) $x - x^2 = 4$ | f) $x^2 + 5x + 3 = 0$ |

3. The width of a rectangle is five less than its length. Find the dimensions of the rectangle if its area is 140 square meters. Give your dimensions to the nearest hundredth of a meter.

1.5 FURTHER USES OF THE QUADRATIC FORMULA

Suppose we need to solve the equation $2x - 6\sqrt{x} = 3$. At first glance, this problem may not look like a quadratic equation since there is no x^2 term. However, note that this equation can be rewritten as

$$2(\sqrt{x})^2 - 6\sqrt{x} - 3 = 0,$$

which is a quadratic equation in the variable \sqrt{x} . The quadratic formula then tells us that

$$\sqrt{x} = \frac{6 \pm \sqrt{36 - 4(2)(-3)}}{2(2)} = \frac{6 \pm \sqrt{60}}{2(2)} = \frac{3 \pm \sqrt{15}}{2}.$$

At this stage, we need to be careful. Recall that \sqrt{x} represents the positive square root of x , that is, the number \sqrt{x} must be positive. It follows that the only allowed choice for the solution is the one with the plus sign since the minus sign gives a negative number. Thus

$$x = \left(\frac{3 + \sqrt{15}}{2}\right)^2 = \frac{9 + 6\sqrt{15} + 15}{4} = 6 + \frac{3}{2}\sqrt{15}.$$

As a second example, consider the following problem.

$$\begin{array}{ll} 2x^4 = 10x^2 - 7 & \text{given problem} \\ 2x^4 - 10x^2 + 7 = 0 & \text{set equal to 0} \\ 2(x^2)^2 - 10(x^2) + 7 = 0 & \text{quadratic in } x^2 \\ x^2 = \frac{10 \pm \sqrt{10^2 - 4(2)(7)}}{2(2)} & \text{use quadratic formula} \\ x^2 = \frac{10 \pm \sqrt{44}}{4} & \text{simplify} \\ x^2 = \frac{5 \pm \sqrt{11}}{2} & \text{simplify} \end{array}$$

Since both of the numbers $5 \pm \sqrt{11}$ are positive, we find four solutions to our equation, namely

$$x = \pm\sqrt{\frac{5 - \sqrt{11}}{2}} \quad \text{and} \quad x = \pm\sqrt{\frac{5 + \sqrt{11}}{2}}.$$

These numbers look rather messy, but a calculator can be used to find decimal approximations if necessary. In this case, the solutions to the nearest thousandth are $x = \pm 0.917$ and $x = \pm 2.039$.

Note: These sorts of problems may be too abstract for most of the students.

Exercises

1. Solve each of the following equations. Give both the exact solutions and the solutions to the nearest hundredth.

a) $x^4 - 5x^2 - 14 = 0$

b) $x^4 + 6 = 5x^2$

c) $2x^4 - 7x^2 - 20 = 0$

d) $2x - 3\sqrt{x} = 10$

e) $x - 5\sqrt{x} - 3 = 0$

f) $x^{2/3} - 6x^{1/3} = 16$

g) $2x^{2/3} = 12 - 7x^{1/3}$

h) $x^6 - 15 = 2x^3$

i) $2x^6 = 3x^3 + 1$

1.6 SOLVING GENERAL PROBLEMS INVOLVING QUADRATIC EQUATIONS

We have considered three different ways to solve quadratic equations: factoring, completing the square, and using the quadratic formula. While it is true that the quadratic formula will always work and give the correct

10 Chapter 1 Quadratic Equations and Functions

answer, it is not always the best choice for solving a quadratic equation. When given a quadratic equation, it is best to try the following steps:

- i. Put the equation in the form $ax^2 + bx + c = 0$. If a , b , and c have a common factor, divide through by this factor to keep the numbers as small as possible. If fractions are involved, multiply through by an appropriate constant to eliminate them.
- ii. After performing step (i), make an attempt to see if the quadratic can be factored. If it can be factored, the problem is easily solved. If, however, you cannot find any factors, move on to step (iii).
- iii. If the coefficient of x^2 is one and the coefficient of x is even, then completing the square is a viable option. You should be able to carry out the details quite quickly.
- iv. If you reach this stage of the process, it is probably best to use the quadratic formula.

We illustrate this scheme with several examples. Study them carefully.

$$\begin{array}{ll} 16x^2 = 56x + 120 & \text{given problem} \\ 16x^2 - 56x - 120 = 0 & \text{set equal to 0} \\ 2x^2 - 7x - 15 = 0 & \text{divide through by 8} \\ (2x + 3)(x - 5) = 0 & \text{factor} \\ x = -3/2, 5 & \text{use the zero product property} \end{array}$$

First of all, notice how dividing through by 8 makes the problem much less intimidating. Although the factors were not completely obvious in this case, the fact that 2 can only be factored in one way and 15 has only a few divisors means that it does take long to see if some combination of these factors will give us the desired trinomial. The next example involves fractions.

$$\begin{array}{ll} \frac{1}{12}x^2 + \frac{1}{6}x = \frac{7}{3} & \text{given problem} \\ x^2 + 2x = 28 & \text{eliminate the fractions} \\ x^2 + 2x + 1 = 29 & \text{complete the square} \\ (x + 1)^2 = 29 & \text{simplify} \\ x = -1 \pm \sqrt{29} & \text{solve for } x \end{array}$$

Once again, multiplying by a constant (12 in this case) makes the equation much easier to solve. Since the coefficient of x^2 is one and the coefficient of x is even, we opted to complete the square. (Note that we did not bother to set the equation equal to 0.) For a final example, we consider the following equation.

$$\begin{array}{ll} -5x^2 + 7x = -13 & \text{given problem} \\ -5x^2 + 7x + 13 = 0 & \text{set equal to 0} \end{array}$$

$$\begin{aligned}
 5x^2 - 7x - 13 &= 0 && \text{make } x^2 \text{ coefficient positive} \\
 x &= \frac{7 \pm \sqrt{7^2 - 4(5)(-13)}}{2(5)} && \text{use quadratic formula} \\
 x &= \frac{7 \pm \sqrt{309}}{10} && \text{simplify}
 \end{aligned}$$

Since the numbers 5 and 13 are prime, there are very few factors to consider and no combination of them can generate a middle term of $7x$. Completing the square would lead to lots of fractions so we avoid this technique as well. Although it is not necessary to make the x^2 coefficient positive, it generally simplifies the form of the final answer.

Exercises

1. Solve each of the following equations. Give both the exact solutions and the solutions to the nearest hundredth.

a) $x^2 - 33x = 70$	b) $x^2 = 14x + 11$	c) $x^2 - 3x - 1 = 0$
d) $4x^2 = 24x + 40$	e) $3x^2 = 14x + 5$	f) $4x^2 + 10x = 24$
g) $x^2 = 10 - 7x$	h) $\frac{1}{6}x^2 + \frac{2}{3}x + \frac{1}{6} = 0$	i) $4x^2 + 216 = 60$
j) $(x + 4)^2 = 2x + 14$	k) $2(x^2 + 7) = 11x$	l) $x^2 + \frac{4}{3}x = 7$
m) $(2x + 3)^2 = 45 + (x - 3)^2$	n) $16x^2 - 80x = 192$	o) $3x^2 + 18x = 48$

1.7 FINDING QUADRATIC EQUATIONS

Given a quadratic equation, we now have several methods to find the solutions: factoring, completing the square, and the quadratic formula. Suppose we reverse the process and consider the problem of finding a quadratic equation given its solutions. In some cases, this is quite easy. For example, if the solutions are $x = -3$ and $x = 4$, then one possibility for the equation would be

$$(x + 3)(x - 4) = 0 \quad \text{or} \quad x^2 - x - 12 = 0.$$

Knowing the solutions ($x = -3$ and $x = 4$) tells us what the factors ($x + 3$ and $x - 4$) should be. Note that the quadratic equations

$$5x^2 - 5x - 60 = 0 \quad \text{and} \quad 0.1x^2 - 0.1x - 1.2 = 0$$

also have $x = -3$ and $x = 4$ as solutions. This shows that there is more than one correct quadratic equation for these types of problems.

We consider two other examples to illustrate various ways to solve these kinds of problems. Suppose we want to find a quadratic equation with solutions $x = 5 \pm 2\sqrt{3}$. The following steps solve this problem:

$$x = 5 \pm 2\sqrt{3} \quad \text{given solutions}$$

$$\begin{array}{ll}
 x - 5 = \pm 2\sqrt{3} & \text{isolate the } \pm \text{ term} \\
 (x - 5)^2 = 12 & \text{square both sides} \\
 x^2 - 10x + 25 = 12 & \text{multiply out} \\
 x^2 - 10x + 13 = 0 & \text{set equal to 0}
 \end{array}$$

The desired equation is thus $x^2 - 10x + 13 = 0$.

For a second example, suppose that the roots of a quadratic equation are $x = \frac{1}{3} \pm \frac{1}{2}\sqrt{5}$. To find the quadratic equation, we perform some algebra:

$$\begin{array}{ll}
 x = \frac{1}{3} \pm \frac{1}{2}\sqrt{5} & \text{given solutions} \\
 6x = 2 \pm 3\sqrt{5} & \text{eliminate the fractions} \\
 6x - 2 = \pm 3\sqrt{5} & \text{isolate the } \pm \text{ term} \\
 (6x - 2)^2 = 45 & \text{square both sides} \\
 36x^2 - 24x + 4 = 45 & \text{multiply out} \\
 36x^2 - 24x - 41 = 0 & \text{set equal to 0}
 \end{array}$$

A quadratic equation with the given solutions is thus $36x^2 - 24x - 41 = 0$.

We can look at this problem in a general setting. Suppose we want to find a quadratic equation with solutions $x = r$ and $x = s$, where r and s are given real (or complex) numbers. One possibility for a quadratic equation with these solutions is

$$(x - r)(x - s) = 0 \quad \text{or} \quad x^2 - (r + s)x + rs = 0.$$

Note that the product of the roots gives the constant term and that the negative of the sum of the roots gives the coefficient for the linear term. However, any equation of the form

$$a(x - r)(x - s) = 0 \quad \text{or} \quad ax^2 - a(r + s)x + ars = 0$$

for any nonzero real number a also has solutions $x = r$ and $x = s$. If we compare the equations

$$ax^2 + bx + c = 0 \quad \text{and} \quad ax^2 - a(r + s)x + ars = 0,$$

we find that $r + s = -b/a$ and $rs = c/a$. In other words, we know the sum and product of the solutions to $ax^2 + bx + c = 0$ without even solving the equation. To illustrate this fact, consider the quadratic equation $5x^2 - 17x - 40 = 0$. If x_1 and x_2 are the solutions to this equation, then $x_1 + x_2 = 17/5$ and $x_1x_2 = -8$; be sure you see how these numbers are determined.

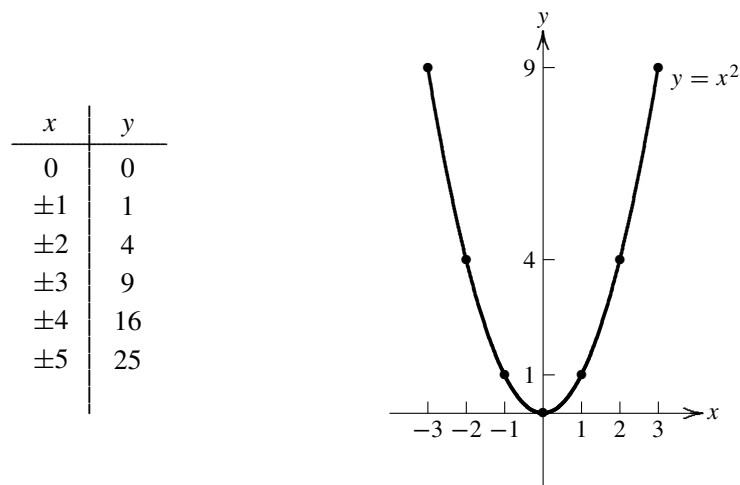
Exercises

1. Find a quadratic equation that has the given solutions.

- a) $x = -2, x = 7$ b) $x = 12, x = 15$ c) $x = -8, x = -3$
 d) $x = \frac{1}{4}, x = 5$ e) $x = -\frac{7}{2}, x = \frac{8}{3}$ f) $x = \frac{5}{6}, x = \frac{11}{4}$
2. Find a quadratic equation that has the given solutions.
 a) $x = -2 \pm \sqrt{7}$ b) $x = 4 \pm 3\sqrt{6}$ c) $x = 10 \pm \sqrt{2}$
 d) $x = \frac{1}{4}, x = 5$ e) $x = -\frac{7}{2}, x = \frac{8}{3}$ f) $x = \frac{5}{6}, x = \frac{11}{4}$
3. Find the sum and product of the roots to the given quadratic equation; you do not need to find the solution.
 a) $x^2 - 4x - 61 = 0$ b) $x^2 + 5x - 12 = 0$ c) $x^2 = 12.8x - 3.1$
 d) $4x^2 - 7x - 31 = 0$ e) $-3x^2 + 14x + 5 = 0$ f) $5x^2 = 18x + 7$
4. Sally solved the equation $2x^2 - 4.32x + 1.73 = 0$ and obtained $x \approx 0.42$ and $x \approx 1.55$ as the approximate solutions. How can you quickly convince her that she made an error?

1.8 GRAPHS OF QUADRATIC FUNCTIONS

Suppose we want to sketch a graph of the equation $y = x^2$. Recall that the graph of this functional relationship represents the set of all ordered pairs (x, y) such that $y = x^2$. To get an idea of what this graph might look like, we can make a table of values and plot the corresponding points on an x - y coordinate grid. Assuming a regular pattern, we can then connect the dots with a smooth curve. If necessary, we can plot many more points or ask a calculator to do the work for us. A brief table of values and a portion of the graph of $y = x^2$ is given below.

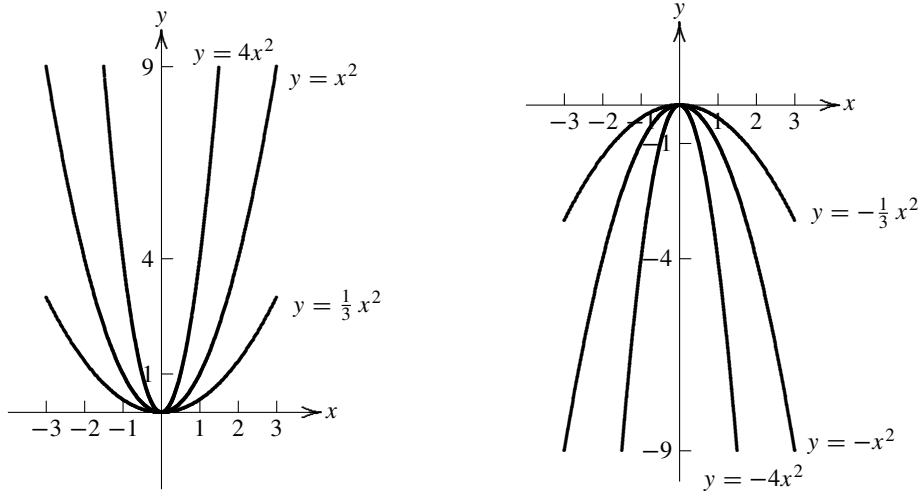


The resulting U -shaped graph is known as a parabola. The point $(0, 0)$ at the base of the U is called the vertex of the parabola and the y -axis (that is, the vertical line $x = 0$) is a line of symmetry of the parabola.

Recall from our work with the absolute value function $y = |x|$ that once we know the graph of $y = |x|$ (it is shaped like a \vee), we can easily find the graph of any function of the form $y = a|x - h| + k$, where a , h , and k are constants. The same thing happens for the graphs of quadratic functions; they all have a U -shape.

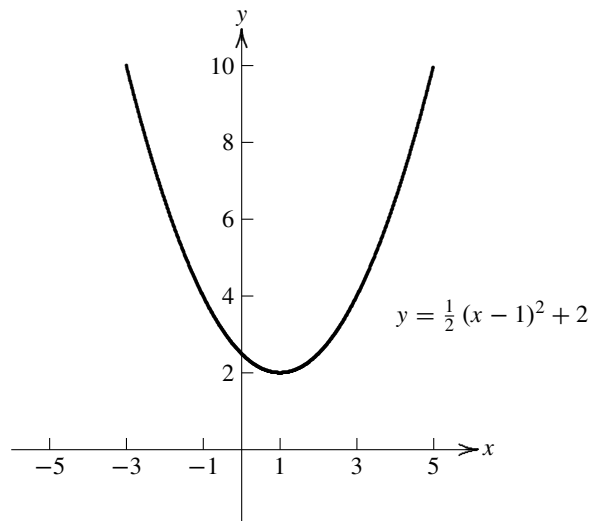
Let's first consider what happens when we look at the graph of $y = ax^2$, where a is a nonzero constant. By plotting a few points, we can easily sketch some simple graphs of this type. The graphs below show the cases

$a = 1$, $a = 4$, and $a = 1/3$ on the coordinate system on the left. The graphs are shown on the same set of axes for comparison purposes. As a gets bigger, the U -shape becomes narrower or, to say it another way, as a gets smaller, the U -shape becomes wider.



The three graphs that are sketched on the coordinate system on the right show what happens when a is negative; the graph simply turns over and the U -shape is inverted. Notice that the vertex in this case is the high point on the graph instead of the low point.

The general graph of the quadratic function $y = a(x - h)^2 + k$ is just the graph of $y = ax^2$ shifted (or translated) from the origin $(0, 0)$ to the point (h, k) . The vertex of the parabola is now (h, k) , the line of symmetry of the parabola is $x = h$, and the parabola opens up when a is positive and opens down when a is negative. For example the graph of $y = \frac{1}{2}(x - 1)^2 + 2$ is just the graph of $y = \frac{1}{2}x^2$ shifted to the right one unit and up two units. The graph of this function is shown below.



Note that the graph of this parabola crosses the y -axis at the point $(0, 2.5)$ and that the graph does not cross the x -axis; there are no x -intercepts.

As a final example, consider the quadratic function $y = 800 - 4(x - 30)^2$. The vertex of the corresponding parabola has coordinates $(30, 800)$ and the line of symmetry of the parabola is $x = 30$. The y -intercept occurs when $x = 0$. Substituting this value of x into the equation gives $y = 800 - 4 \cdot 900 = -2800$ so the y -intercept is $(0, -2800)$. To find the x -intercepts, we need to solve the equation $800 - 4(x - 30)^2 = 0$. We find that

$$4(x - 30)^2 = 800 \Rightarrow (x - 30)^2 = 200 \Rightarrow x - 30 = \pm\sqrt{200} \Rightarrow x = 30 \pm 10\sqrt{2}.$$

The x -intercepts are thus approximately $(44.14, 0)$ and $(15.86, 0)$.

Exercises

- Sketch the graphs of each of the following functions. Find the y -intercept in each case and the x -intercepts when appropriate.

a) $y = \frac{1}{5}x^2$	b) $y = -3x^2$	c) $y = x^2 - 4$
d) $y = 2x^2 + 3$	e) $y = 12 - x^2$	f) $y = \frac{1}{3}(x - 2)^2 + 4$
g) $y = 2(x + 1)^2 - 6$	h) $y = -(x - 3)^2 - 4$	i) $y = -2(x - 1)^2 + 18$
- Give the (x, y) coordinates of the vertex of each parabola and the equation of the line of symmetry. Find the y -intercept in each case and the x -intercepts when appropriate.

a) $y = 2(x - 18)^2 - 60$	b) $y = -4(x - 300)^2 + 1600$	c) $y = 800 - 2(x - 75)^2$
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- Suppose that $(-2, 4)$ and $(14, 4)$ are two points on the graph of a parabola. What is the equation of the axis of symmetry for the parabola?

1.9 GRAPHS OF QUADRATIC FUNCTIONS, CONTINUED

In the last section, we showed how the graph of $y = a(x - h)^2 + k$ is related to the graph of $y = ax^2$. However, not every quadratic function is given in this convenient form. How do you find the vertex of a parabola when the equation is given in the form $y = ax^2 + bx + c$? It turns out that the technique of completing the square is quite useful for these problems. To find the vertex of the parabola given by the equation $y = x^2 - 2x + 4$, perform the following steps:

$$\begin{aligned} y &= x^2 - 2x + 4 && \text{given function} \\ y &= x^2 - 2x + 1 + 3 && \text{complete the square, } (-2/2)^2 = 1 \\ y &= (x - 1)^2 + 3 && \text{simplify} \end{aligned}$$

It can now be seen that the vertex is $(1, 3)$. Note how the number 4 was written as $1 + 3$ so that the 1 could be used to complete the square. To be sure the ideas are clear, here is a similar example.

$$\begin{aligned} y &= x^2 + 8x - 7 && \text{given function} \\ y &= x^2 + 8x + 16 - 23 && \text{complete the square, } (8/2)^2 = 16 \\ y &= (x + 4)^2 - 23 && \text{simplify} \end{aligned}$$

16 Chapter 1 Quadratic Equations and Functions

The vertex of the parabola given by the equation $y = x^2 + 8x - 7$ is $(-4, -23)$. Once again, note that the constant term -7 has been written as $16 - 23$. We are not doing something to both sides of the equation; rather we are rewriting a number in an equivalent but more helpful way.

When the leading coefficient of the x^2 term is not 1, extra care needs to be taken. The first step is to factor a constant out of the x terms so that the coefficient of x^2 is 1. When we complete the square on the factored expression, it is important to realize that we are only working with one side of an equation. This means that terms that are inserted must also be removed in order to keep the function unchanged. Carefully study the following example:

$$\begin{aligned}y &= 3x^2 - 12x + 17 && \text{given function} \\y &= 3(x^2 - 4x) + 17 && \text{factor a constant from the } x \text{ terms} \\y &= 3(x^2 - 4x + 4) - 12 + 17 && \text{complete the square, } (-4/2)^2 = 4 \\y &= 3(x - 2)^2 + 5 && \text{simplify}\end{aligned}$$

Note that $3(4) - 12 = 0$ so we did not change the function. The vertex of the parabola given by $y = 3x^2 - 12x + 17$ is thus $(2, 5)$. When negative numbers are involved, we need to be even more careful.

$$\begin{aligned}y &= 21 - 16x - 2x^2 && \text{given function} \\y &= -2(x^2 + 8x) + 21 && \text{factor a constant from the } x \text{ terms} \\y &= -2(x^2 + 8x + 16) + 32 + 21 && \text{complete the square, } (8/2)^2 = 16 \\y &= -2(x + 4)^2 + 53 && \text{simplify}\end{aligned}$$

In this case, we see that $(-2)(16) + 32 = 0$ so that the equation remains unchanged. It follows that the vertex of the parabola given by $y = 21 - 16x - 2x^2$ is $(-4, 53)$.

Exercises

1. Find the (x, y) coordinates of the vertex of each parabola. State whether or not the vertex corresponds to the highest point on the graph or the lowest point on the graph.

a) $y = x^2 - 4x + 15$

b) $y = -x^2 + 2x - 5$

c) $y = x^2 + 6x - 4$

d) $y = 2x^2 + 12x - 3$

e) $y = 3x^2 - 24x + 48$

f) $y = 5x^2 + 10x - 24$

g) $y = \frac{1}{3}x^2 - 6x + 5$

h) $y = 28 - 4x - x^2$

i) $y = 25 + 5x - \frac{1}{2}x^2$

j) $y = \frac{1}{4}x^2 - 2x + 15$

k) $y = (4 + x)(5 - x)$

l) $y = (2x + 1)(x - 8)$

1.10 APPLICATIONS OF QUADRATIC FUNCTIONS

One of the applications of quadratic functions concerns motion in a gravitational field. On the earth, if an object is launched upward from a height of p feet with an initial velocity v feet per second, then its height at any time t

(in seconds) is given by the function

$$h(t) = -16t^2 + vt + p.$$

(It should be noted that this formula is only correct if air resistance is ignored. In other words, it gives accurate results for baseballs and rocks but would not be accurate for leaves or parachutes.) Since the graph of the function $h(t)$ is a parabola, we can use the ideas in this chapter to answer questions about the motion of the object.

For an example, suppose that a ball is thrown straight upward from a height of 60 feet with an initial velocity of 80 feet per second. Using the above formula, we find that the height of the ball (in feet) at any time t (seconds) is given by $h(t) = -16t^2 + 80t + 60$. This function can then be used to determine such things as (a) the maximum height of the ball, (b) the length of time the ball is in the air, and (c) the amount of time the ball spends above a height of 100 feet.

To answer (a), we express the function $h(t)$ in vertex form:

$$h(t) = -16t^2 + 80t + 60 = -16\left(t^2 - 5t + \frac{25}{4}\right) + 160 = -16\left(t - \frac{5}{2}\right)^2 + 160.$$

We thus see that the graph of $h(t)$ is a parabola opening downward and its vertex is $(2.5, 160)$. The maximum height of the ball is 160 feet and this occurs after 2.5 seconds.

To answer (b), we know that the ball is on the ground when $h(t) = 0$. We could use the quadratic formula to solve this equation, but since we have already completed the square, we will use this form:

$$h(t) = 0 \Rightarrow -16\left(t - \frac{5}{2}\right)^2 + 160 = 0 \Rightarrow \left(t - \frac{5}{2}\right)^2 = 10 \Rightarrow t = \frac{5}{2} \pm \sqrt{10}.$$

Since the time we are looking for is positive, we choose the solution that has the plus sign. Hence, the ball is in the air for a total of $2.5 + \sqrt{10} \approx 5.662$ seconds.

Turning now to (c), we need to solve the equation $h(t) = 100$. We find that

$$h(t) = 100 \Rightarrow -16t^2 + 80t + 60 = 100 \Rightarrow -16t^2 + 80t - 40 = 0 \Rightarrow 2t^2 - 10t + 5 = 0.$$

Note how we have cancelled a common factor. This time we will use the quadratic formula to solve the equation:

$$t = \frac{10 \pm \sqrt{(-10)^2 - 4(2)(5)}}{4} = \frac{10 \pm \sqrt{60}}{4} = \frac{5 \pm \sqrt{15}}{2}.$$

Thus the ball is at a height of 100 feet at the times

$$t_1 = \frac{5 - \sqrt{15}}{2} \quad \text{and} \quad t_2 = \frac{5 + \sqrt{15}}{2}.$$

The total time the ball is above 100 feet is thus $t_2 - t_1 = \sqrt{15} \approx 3.873$ seconds.

Exercises

1. Suppose that a ball is thrown straight upward from a height of 160 feet with an initial velocity of 80 feet per second. Find (a) the maximum height of the ball, (b) the length of time the ball is in the air, and (c) the amount of time the ball spends above a height of 200 feet.

2. Suppose that a ball is thrown straight upward from a height of 40 feet with an initial velocity of 120 feet per second. Find (a) the maximum height of the ball, (b) the length of time the ball is in the air, and (c) the amount of time the ball spends above a height of 60 feet.
3. A toy rocket is shot straight upward from the ground with an initial velocity of 180 feet per second. How high does it go and how long is it in the air?
4. The Sears Tower in Chicago has a height of 1450 feet. How long does it take for an object dropped from the top of this building to hit the ground?
5. A kangaroo bounds into the air with an initial velocity of 20 feet per second. How long is the kangaroo in the air?
6. A ball thrown upward from the ground reaches a maximum height of 225 feet. What was its initial velocity?
7. A hot air balloon is ascending at the rate of 40 ft/sec at a height 300 ft above the ground when a brick is dropped from the balloon. How long does it take for the brick to reach the ground? Note that the brick initially goes up a bit before falling to the ground.
8. A ball thrown straight up from a height of 20 ft takes 2 sec to reach a height of 100 ft. What was its initial speed? How much higher will the ball go?
9. Suppose that a car can decelerate at a rate of a ft/sec². If the initial velocity of the car is v_0 ft/sec, then the velocity of the car in ft/sec and the distance in ft traveled by the car after the brakes are applied are given by

$$v(t) = -at + v_0 \quad \text{and} \quad d(t) = -\frac{a}{2}t^2 + v_0t$$

respectively, where time t is measured in seconds. For some perspective on these problems, the reader should verify that 60 miles per hour is equivalent to 88 feet per second.

- a) A certain car is able to brake with a deceleration of 20 ft/sec². How long does such a car take to come to a stop if it is initially traveling at 88 ft/sec? What is the distance traveled during the braking process? Answer these questions if the initial velocity is 120 ft/sec.
- b) A certain car is able to brake with a deceleration of a ft/sec². Find a value for a so that a car traveling at 100 ft/sec can come to a stop after the car has traveled 200 ft. After finding a , determine how long it takes for the car to stop.
10. A line segment of length 10 units is cut into two pieces in such a way that the ratio of the whole segment to the larger piece equals the ratio of the larger piece to the shorter piece. Find the lengths of the two pieces and the ratio of the larger piece to the smaller piece.
11. A calculator company has been making 1500 programmable calculators monthly at a total cost of \$76750 and selling them at \$57 each; thus making a monthly profit of \$8750. An analysis of the company's operations shows that the cost of producing x calculators in a month is $C(x) = 0.003x^2 + 30x + 25000$, and that the number of calculators that can be sold monthly for p dollars each is $x = 30000 - 500p$. How many calculators should the company make, and at what price should it sell them, in order to maximize its monthly profit?
12. Consider the trapezoid with vertices at $(0, 0)$, $(0, 2)$, $(4, 2)$, and $(6, 0)$. Find a horizontal line that divides the trapezoid into two parts of equal area.

1.11 GEOMETRIC DEFINITION OF A PARABOLA

Recall that a circle is a curve in two dimensions that represents the set of all points equidistant from a given fixed point. The fixed point is the center of the circle and the distance is known as the radius. Suppose we want to find an equation for the circle centered at $(0, 0)$ with radius 3. Using the distance formula, we find that any point (x, y) on this circle must satisfy the equation

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 3 \quad \text{or} \quad x^2 + y^2 = 9.$$

For a slightly more complicated example, the circle centered at $(-2, 5)$ with radius 8 has equation

$$\sqrt{(x + 2)^2 + (y - 5)^2} = 8 \quad \text{or} \quad (x + 2)^2 + (y - 5)^2 = 64.$$

In general, an equation for a circle centered at (h, k) with radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

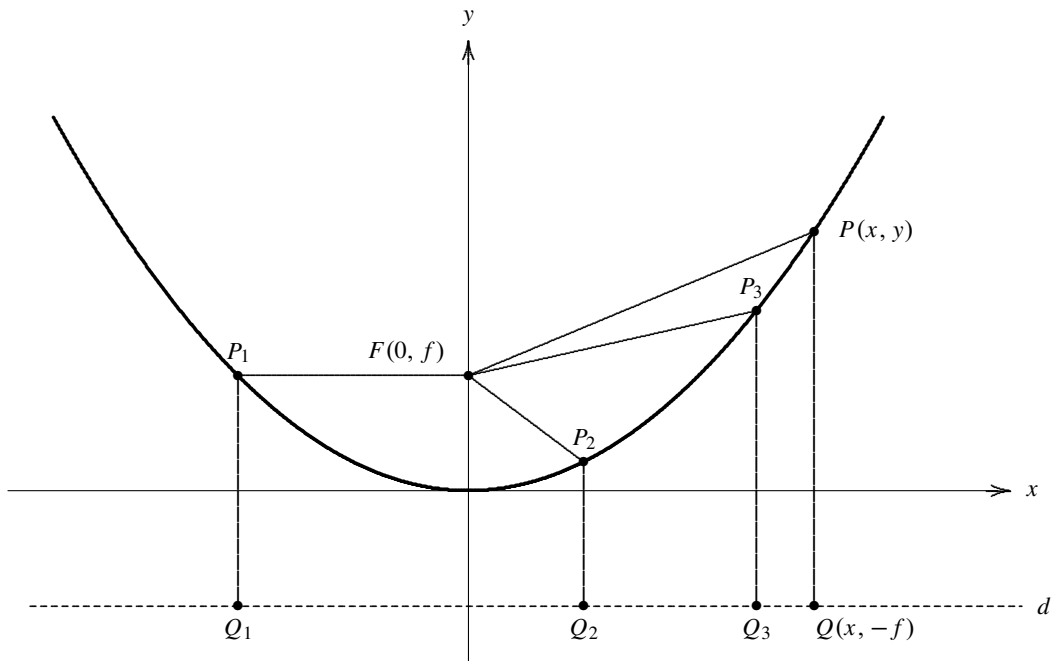
This means that the circle with radius r and center (h, k) is the set of all points (x, y) that satisfy the above equation.

Since we have learned how to complete the square, we can reverse this process and find the center and radius of a circle given by an equation in a different form. Carefully ponder the following set of equations:

$$\begin{aligned} x^2 + 4x + y^2 - 2y &= 11 && \text{given problem} \\ x^2 + 4x + 4 + y^2 - 2y + 1 &= 11 + 4 + 1 && \text{complete the square for } x \text{ and } y \\ (x + 2)^2 + (y - 1)^2 &= 16 && \text{simplify} \end{aligned}$$

It follows that the equation represents a circle centered at $(-2, 1)$ with radius 4.

Switching gears a bit, consider the curve in two dimensions that represents the set of all points that are exactly the same distance from a given fixed point and a given line. The fixed point is known as the focus and the fixed line is called the directrix. In the figure below, point F is the focus, line d is the directrix, and the dark curve is the set of all points equidistant from F and d .



For example, the points $P_1, P_2,$ and P_3 are on the curve since $\overline{FP_1} = \overline{P_1Q_1}, \overline{FP_2} = \overline{P_2Q_2},$ and $\overline{FP_3} = \overline{P_3Q_3}.$

Suppose we let the coordinates of F be $(0, f)$ and the equation of the directrix be $y = -f.$ In order for a point P with coordinates (x, y) to be on the curve, we must have the distance from F to P be the same as the

distance from P to line d . Using the distance formula for the former and the vertical distance for the latter, we need

$$\sqrt{(x - 0)^2 + (y - f)^2} = y + f$$

Squaring both sides and simplifying yields

$$x^2 + (y - f)^2 = (y + f)^2 \Leftrightarrow x^2 + y^2 - 2fy + f^2 = y^2 + 2fy + f^2 \Leftrightarrow y = \frac{1}{4f} x^2.$$

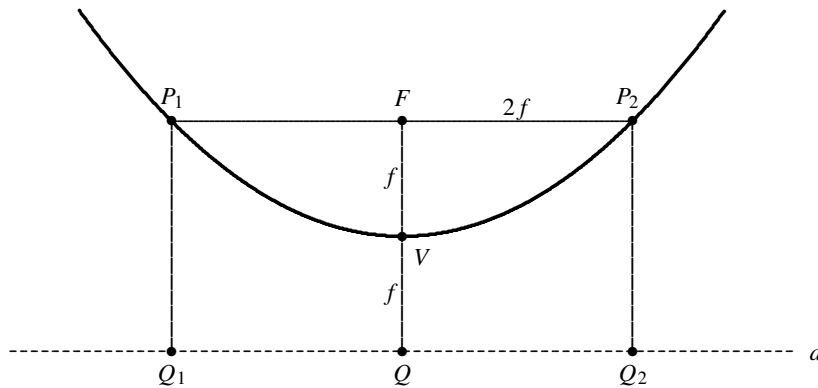
We thus see that the curve formed by these conditions is a parabola. Note that the number f represents the distance from the focus to the vertex of the parabola.

Comparing the two forms of a parabola, namely $y = \frac{1}{4f} x^2$ and $y = ax^2$, we see that the focal distance f is equal to $1/4a$. This means that we can find the focus and directrix of a parabola from the vertex form of the quadratic function. For example, suppose we want to find the focus and directrix for the parabola given by the equation $y = \frac{1}{2} x^2 - 2x + 5$. The first step is to write the equation in vertex form:

$$y = \frac{1}{2} x^2 - 2x + 5 = \frac{1}{2} (x^2 - 4x + 4) - 2 + 5 = \frac{1}{2} (x - 2)^2 + 3.$$

The vertex is $(2, 3)$ and (since the value of a is $1/2$) the focal distance is $1/2$. Since the parabola opens up, the focus is the point $(2, 3.5)$ and the directrix is the line $y = 2.5$.

Although it is not a pleasing sounding term for modern ears, the latus rectum of a parabola is the chord parallel to the directrix and passing through the focus. In the figure below, the segment $\overline{P_1 P_2}$ is the latus rectum of the parabola.



This term is a composed of two Latin words; latus is the Latin term for side and rectum is the Latin term for straight. Since we have denoted the length of \overline{FV} as f , it follows (from the definition of the parabola) that the length of \overline{VQ} is also f . Consequently, the length of $\overline{P_2 Q_2}$ is $2f$ and (by the definition of the parabola again) the length of $\overline{FP_2}$ is $2f$. We have thus shown that the length of the latus rectum of a parabola is $4f$. Referring to the comments made in the previous paragraph, we see that the length of the latus rectum is $1/a$ when the parabola is expressed in vertex form.

Exercises

- Find an equation for a circle with the given center and radius.

a) $(0, 0)$, $r = 7$	b) $(0, 0)$, $r = 20$	c) $(1, 2)$, $r = 3$
d) $(5, 3)$, $r = 6$	e) $(-4, 2)$, $r = 1$	f) $(-3, -2)$, $r = 2\sqrt{3}$
- Find the center and radius of the circle with the given equation.

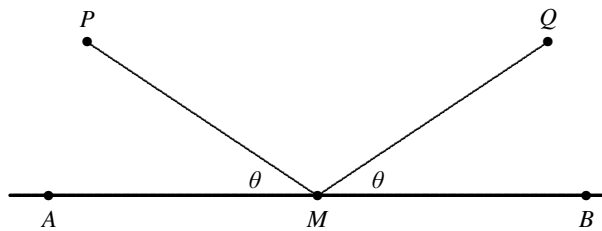
a) $x^2 - 2x + y^2 = 24$	b) $x^2 - 6x + y^2 - 4y = 36$
c) $x^2 + 4x + y^2 - 10y = 71$	d) $x^2 + 8x + y^2 + 12y = 28$
e) $x^2 - 3x + y^2 - 2y = 5$	f) $x^2 + 5x + y^2 + y = 8$
- Find an equation for the parabola with the given focus and directrix.

a) $(0, 4)$, $y = -4$	b) $(0, \frac{1}{2})$, $y = -\frac{1}{2}$
c) $(4, 5)$, $y = -2$	d) $(1, 3)$, $y = 1$
- Find the focus and directrix for the given parabola. Also determine the length of the latus rectum.

a) $y = \frac{1}{20}x^2$	b) $y = 3x^2$
c) $y = \frac{1}{8}x^2 - 2x$	d) $y = \frac{1}{4}x^2 + 3x + 8$
e) $y = x^2 - 4x + 5$	f) $y = 2x^2 - 12x - 7$

1.12 REFLECTION PROPERTY OF PARABOLAS

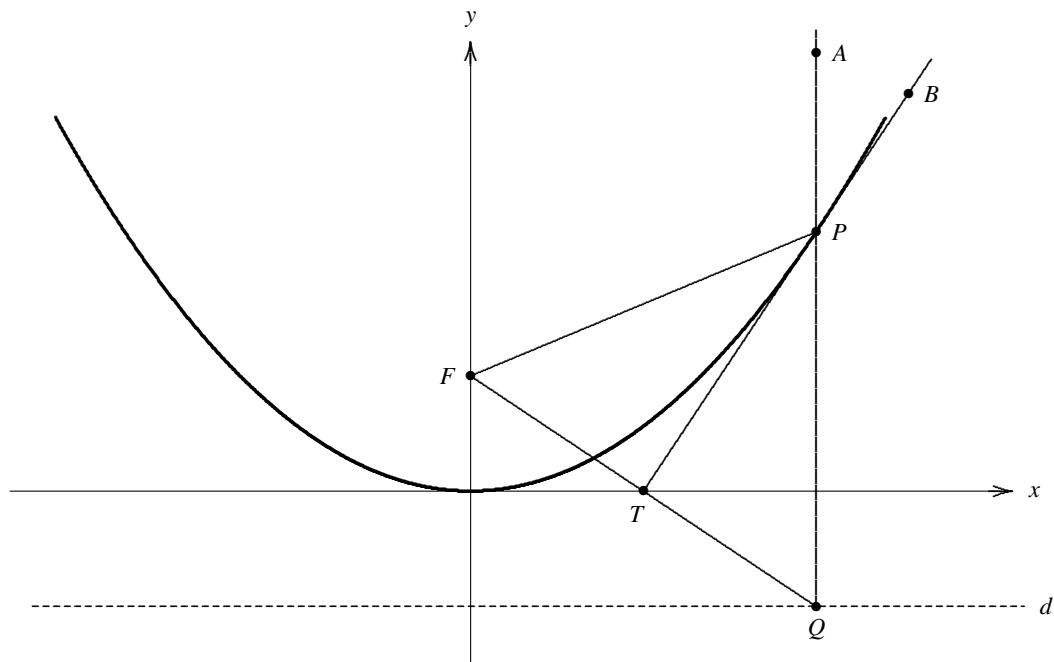
Parabolas have an interesting reflection property that makes them useful for several practical purposes, including headlights in cars and lenses in telescopes. You may recall from physics that when light is reflected off of a surface, the angle of incidence equals the angle of reflection. In the figure below, the light source is at P and the light contacts the mirror (represented by the dark line) at M . The reflected light passes through the point Q in such a way that angles $\angle PMA$ and $\angle QMB$ are equal; these angles are marked θ in the figure.



In this context, $\angle PMA$ is called the angle of incidence and $\angle QMB$ is called the angle of reflection.

Since a parabola is a curved surface rather than a straight line, a bit more effort is required to determine how a light beam will reflect off of it. If you zoom in on a curve, you will notice that it begins to look like a straight line. This line is known as the tangent line to the curve; over a short distance, it passes through just one point on the curve and resembles the curve.

need some more work here to explain tangent line for a parabola



Suppose we let the coordinates of F be $(0, f)$ and the coordinates of P be (a, b) . Then the coordinates of T are $(a/2, 0)$ by properties of tangent lines and the coordinates of Q are $(a, -f)$. It follows easily that $\overline{FT} = \overline{QT}$. By the definition of the parabola, we know that $\overline{PF} = \overline{PQ}$. Therefore, triangles $\triangle FPT$ is congruent to $\triangle QPT$ by SSS. It follows that $\angle QPT = \angle FPT$ since they are corresponding angles in these congruent triangles. Noting that $\angle APB = \angle QPT$ since they are vertical angles, we conclude that $\angle APB = \angle FPT$. This means that a ray of light starting at the focus F will reflect off of the parabola at point P in a direction parallel to the axis of symmetry of the parabola.

and thus ...

1.13 QUADRATIC INEQUALITIES

Exercises