Intermediate Value Theorem: If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and v is any number between $f(a)$ and $f(b)$, then there is a number c in (a,b) such that $f(c) = v$.

Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f has both a maximum output and a minimum output on $[a,b]$. In other words, there exist numbers $c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.

Mean Value Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Fundamental Theorem of Calculus: If f is continuous on an interval $[a,b]$ and a function F is defined by $F(x) = \int_a^x f(t)\,dt$ for all x in $[a,b]$, then $F'(x) = f(x)$ for all x in $[a,b]$.

Fundamental Theorem of Calculus: If f is a continuous function defined on an interval $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$ and suppose that $f'' > 0$ on $\mathbb{R}$. Prove that for each real number L, the set $\{x \in \mathbb{R} : f(x) = L\}$ contains at most two points.

2. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $|f'(x)| < 1$ for all $x \in (a,b)$. Prove that f has at most one fixed point.

3. Suppose that f and g are continuous on $[a,b]$ and that $\int_a^b f = \int_a^b g$. Prove that there exists a point $c \in [a,b]$ such that $f(c) = g(c)$.

4. Evaluate $\displaystyle\int_1^3 \frac{6}{\sqrt{x}(1+x)}\,dx$.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a three times differentiable function such that f has at least six distinct positive zeros. Consider the function g defined by

$$g(x) = x^3 f'''(x) - 3x^2 f''(x) + 6x f'(x) - 6f(x).$$

   Prove that g has at least three distinct positive zeros.

6. Evaluate $\displaystyle\int_0^\infty \frac{3x^3 + 7x^2}{(x^3 + x + 1)(x^3 + 4x + 8)}\,dx$.

Problem: Let $f\colon [0,1] \to \mathbb{R}$ be continuous, with $\int_0^1 x(x-1)^2 f(x)\,dx = 0$. Prove that there is a real number $c \in (0,1)$ such that $\int_0^c x^2 f(x)\,dx = c\int_0^c x f(x)\,dx$.

Solution: For each $x \in [0,1]$, let

$$U(x) = \int_0^x (xs - s^2) f(s)\,ds \qquad \text{and} \qquad V(x) = \int_0^x U(t)\,dt.$$

Since the function U is continuous on $[0,1]$, the function V is both continuous and differentiable on $[0,1]$. It is clear that $V(0) = 0$ and, using the given properties of the function f, we find that

$$\begin{aligned}
V(1) &= \int_0^1 U(t)\,dt \\
&= \int_0^1 \int_0^t (ts - s^2) f(s)\,ds\,dt \\
&= \int_0^1 \int_s^1 (ts - s^2) f(s)\,dt\,ds \\
&= \int_0^1 f(s)\Big(\frac{s}{2}\,t^2 - s^2 t\Big)\Big|_s^1\,ds \\
&= \int_0^1 f(s)\Big(\frac{s}{2} - s^2 - \frac{s^3}{2} + s^3\Big)\,ds \\
&= \frac{1}{2}\int_0^1 \Big(s^3 - 2s^2 + s\Big) f(s)\,ds \\
&= \frac{1}{2}\int_0^1 s(s-1)^2 f(s)\,ds \\
&= 0.
\end{aligned}$$

By the Mean Value Theorem, there exists $c \in (0,1)$ such that $V'(c) = 0$. Noting that $V'(c) = U(c)$ since U is a continuous function, we obtain

$$0 = U(c) = \int_0^c (c\,s - s^2) f(s)\,ds = c\int_0^c s f(s)\,ds - \int_0^c s^2 f(s)\,ds.$$

Hence, there is a real number $c \in (0,1)$ such that

$$\int_0^c x^2 f(x)\,dx = c\int_0^c x f(x)\,dx.$$

This completes the proof. ■

Problem: Evaluate $\displaystyle\int_0^1 \frac{(1-x)(\ln x)^2}{1+x^3}\,dx$.

Solution: Using the fact that

$$\int_0^1 x^n (\ln x)^2\,dx = \frac{2}{(n+1)^3}$$

for all nonnegative integers n, we obtain

$$\begin{aligned}
\int_0^1 \frac{(1-x)(\ln x)^2}{1+x^3}\,dx &= \int_0^1 (1-x)(\ln x)^2 \sum_{k=1}^{\infty}(-x^3)^{k-1}\,dx \\
&= \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 (x^{3k-3}-x^{3k-2})(\ln x)^2\,dx \\
&= \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{2}{(3k-2)^3}-\frac{2}{(3k-1)^3}\right) \\
&= 2\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{(3k-2)^3}-\frac{(-1)^{k-1}}{(3k-1)^3}+\frac{(-1)^{k-1}}{(3k)^3}-\frac{(-1)^{k-1}}{(3k)^3}\right) \\
&= 2\sum_{k=1}^{\infty}\left(\frac{(-1)^{3k-1}}{(3k-2)^3}+\frac{(-1)^{3k}}{(3k-1)^3}+\frac{(-1)^{3k+1}}{(3k)^3}-\frac{(-1)^{k+1}}{27k^3}\right) \\
&= 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}-\frac{2}{27}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3} \\
&= \frac{13}{9}\sum_{k=1}^{\infty}\frac{1}{k^3}.
\end{aligned}$$

The sum of this series is known as Apéry's constant. ∎

Problem: Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(x) + f\Big(1 - \dfrac{1}{x}\Big) = \arctan x$ for all $x \neq 0$. Find $\displaystyle\int_0^1 f(x)\,dx$.

Solution: Replacing x with $(x-1)/x$ two times, we find that

$$\begin{aligned} f(x) + f\Big(\frac{x-1}{x}\Big) &= \arctan x; \\ f\Big(\frac{x-1}{x}\Big) + f\Big(\frac{1}{1-x}\Big) &= \arctan\Big(\frac{x-1}{x}\Big); \\ f\Big(\frac{1}{1-x}\Big) + f(x) &= \arctan\Big(\frac{1}{1-x}\Big); \end{aligned}$$

for all $x \in (0,1)$. It follows that

$$\begin{aligned} f(x) &= \arctan\Big(\frac{1}{1-x}\Big) - f\Big(\frac{1}{1-x}\Big) \\ &= \arctan\Big(\frac{1}{1-x}\Big) - \bigg(\arctan\Big(\frac{x-1}{x}\Big) - f\Big(\frac{x-1}{x}\Big)\bigg) \\ &= \arctan\Big(\frac{1}{1-x}\Big) - \arctan\Big(\frac{x-1}{x}\Big) + \big(\arctan x - f(x)\big) \\ &= \arctan\Big(\frac{1}{1-x}\Big) + \arctan\Big(\frac{1-x}{x}\Big) + \arctan x - f(x); \\ 2f(x) &= \arctan\Big(\frac{1}{1-x}\Big) + \arctan\Big(\frac{1-x}{x}\Big) + \arctan x; \\ 2f(1-x) &= \arctan\Big(\frac{1}{x}\Big) + \arctan\Big(\frac{x}{1-x}\Big) + \arctan(1-x); \\ 2f(x) + 2f(1-x) &= \frac{3\pi}{2}; \end{aligned}$$

for all $x \in (0,1)$. The expression for $f(x)$ in terms of arctangent functions shows that f is continuous and bounded on $(0,1)$ and thus Riemann integrable on $[0,1]$. The value of the integral is

$$\int_0^1 f(x)\,dx = \frac{1}{2}\bigg(\int_0^1 f(x)\,dx + \int_0^1 f(1-x)\,dx\bigg) = \frac{1}{2}\int_0^1 \big(f(x) + f(1-x)\big)\,dx = \frac{3\pi}{8}.$$

This completes the solution. ∎

1. Suppose that n distinct lines are drawn in the plane in such a way that no two lines are parallel and no three lines share a common point. Into how many regions do these n lines divide the plane? Of course, you must provide a proof of your conjecture.

2. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

3. Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize this number. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

4. A class with $2N$ students took a quiz on which the possible scores were integers from 0 to 10. Each of these eleven scores occurred at least once and the average score for the quiz was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group is exactly 7.4.

5. Find all ordered pairs (a, b) of positive integers for which ab divides $a + 6b$.

The set of integers and its properties are at the root of all mathematical disciplines. The symbols $\mathbb{Z}$ and $\mathbb{Z}^+$ will be used to represent the set of integers and the set of positive integers, respectively;

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}.$$

The set of positive integers has an important property that is quite useful in proving some statements that depend on the positive integers. This important property is stated below. (Recall that the symbol $a \in S$ means that a is a member of the set S.)

Principle of Mathematical Induction: If S is a set of positive integers that contains 1 and satisfies the condition " if $k \in S$, then $k+1 \in S$ ", then $S = \mathbb{Z}^+$.

An equivalent way to state this property is the following.

Principle of Mathematical Induction: For each positive integer n, let P_n be a statement that depends on n. If P_1 is true and the conditional statement " if P_k, then P_{k+1} " is valid, then the statement P_n is true for all positive integers n.

The Principle of Mathematical Induction can be compared to a chain reaction. If we know that each event will set off the next (the condition in quotes) and if the first event occurs ($1 \in S$ or P_1 is true), then the entire chain reaction will occur. Perhaps you have seen one of those amazing domino exhibits where thousands of dominoes fall over in interesting patterns. The dominoes must be set up in such a way that each one knocks over the next, and someone must begin the process by pushing over the first domino.

The key step in an induction proof is proving the " if P_k, then P_{k+1} " statement. We make the assumption that P_k is true for some generic positive integer k, then try to use this fact to prove that P_{k+1} is true. It is this part of the proof that may be challenging since the connection between P_k and P_{k+1} may not be immediately clear.

There is a second form of the PMI (Principle of Mathematical Induction). It is sometimes called strong induction (abbreviated PSI for the Principle of Strong Induction) or complete induction. It can be stated in either of the forms below.

If S is a set of positive integers that contains 1 and satisfies the condition "if $1, 2, \ldots, k \in S$, then $k+1 \in S$", then $S = \mathbb{Z}^+$.

For each positive integer n, let P_n be a statement that depends on n. If P_1 is true and the conditional statement " if $P_1, P_2, \ldots, P_k$, then P_{k+1} " is valid, then the statement P_n is true for all positive integers n.

Problem: Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Solution 1: We will use the Principle of Mathematical Induction. Let S be the set of all positive integers n such that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since $1 = (1 \cdot 2 \cdot 3)/6$, it follows that $1 \in S$. Suppose that $k \in S$ for some positive integer k. This means that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

We then have

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k+1}{6}\big(2k^2 + k + 6k + 6\big) \\ &= \frac{k+1}{6}\,(k+2)(2k+3) \\ &= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}, \end{aligned}$$

which indicates that $k+1 \in S$. We have thus shown that "if $k \in S$, then $k+1 \in S$". By the Principle of Mathematical Induction, it follows that $S = \mathbb{Z}^+$. Hence, the equation

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is valid for all positive integers n. ■

Solution 2: The given equation is easily verified for $n = 1$. Suppose that the equation is valid for some positive integer n. Then

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n+1}{6}\big(2n^2 + n + 6n + 6\big) \\ &= \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

showing that the equation is valid for $n+1$ as well. By the Principle of Mathematical Induction, the equation

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is valid for all positive integers n. ■

Problem: For each positive integer n, the number $9^n - 8n - 1$ is a multiple of 64.

Solution: We will use the Principle of Mathematical Induction. Since 0 is a multiple of 64, the statement is valid when $n = 1$. Let k be a positive integer and suppose that $9^k - 8k - 1$ is a multiple of 64. This means that there exists an integer j such that $64j = 9^k - 8k - 1$. We then have

$$\begin{aligned} 9^{k+1} - 8(k+1) - 1 = 9^{k+1} - 8k - 9 &= 9(9^k - 8k - 1) + 64k \\ &= 9 \cdot 64j + 64k = 64(9j + k), \end{aligned}$$

showing that $9^{k+1} - 8(k+1) - 1$ is a multiple of 64. The result now follows by the PMI. ■

Problem: Suppose that $a_1 = 1$, $a_2 = -1/2$, and $a_{n+1} = (a_n + a_{n-1})/2$ for each positive integer $n > 1$. Then $a_n = (-1/2)^{n-1}$ for each positive integer n.

Solution: It is easy to see that the statement is true for $n = 1$ and $n = 2$. (We need to check both of these cases, since these numbers do not fit the general pattern for the generation of terms.) Suppose that $a_n = (-1/2)^{n-1}$ for all the integers $1, 2, \ldots, k$ for some positive integer $k \geq 2$. We must show that $a_{k+1} = (-1/2)^k$. Using the assumption that all the terms up to k satisfy the pattern (all we really need to know is that the pattern is valid for the terms k and $k-1$),

$$a_{k+1} = \frac{1}{2} \cdot \left(a_k + a_{k-1}\right) = \frac{1}{2} \cdot \left[\left(-\frac{1}{2}\right)^{k-1} + \left(-\frac{1}{2}\right)^{k-2}\right] = \frac{1}{2} \cdot \left(-\frac{1}{2}\right)^{k-2} \cdot \left(-\frac{1}{2} + 1\right) = \left(-\frac{1}{2}\right)^{k}.$$

Hence, the equation of interest is valid for $k + 1$. By the Principle of Strong Mathematical Induction, the formula $a_n = (-1/2)^{n-1}$ is valid for all positive integers n. ■

How many pairs of rabbits can be bred in one year from one pair? A certain person places one pair of rabbits in a certain place surrounded on all sides by a wall. We want to know how many pairs can be bred from that pair in one year, assuming it is their nature that each month they give birth to another pair, and in the second month, each new pair can also breed.

The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all $n > 1$.

The Lucas numbers are defined by $\ell_1 = 1$, $\ell_2 = 3$, and $\ell_{n+1} = \ell_n + \ell_{n-1}$ for all $n > 1$.

Let α and $\beta < \alpha$ be the two roots of $x^2 = x + 1$. Note that $\alpha = \phi$ and $\beta = -1/\phi$, where ϕ is the golden mean, and that $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, and $\alpha\beta = -1$.

Using strong induction, it is easy to verify that $f_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$ and $\ell_n = \alpha^n + \beta^n$ for each positive integer n.

The following table lists the first 20 Fibonacci and Lucas numbers. We also use $f_0 = 0$ and $\ell_0 = 2$.

n	f_n	ℓ_n
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123
11	89	199
12	144	322
13	233	521
14	377	843
15	610	1364
16	987	2207
17	1597	3571
18	2584	5778
19	4181	9349
20	6765	15127

1. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for each positive integer n.

2. Prove that $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n\cdot(n+1)} = \dfrac{n}{n+1}$ for each positive integer n.

3. Prove that for each positive integer n, the integer $3^{2n+1} + 2^{n+2}$ is divisible by 7.

4. Prove that for each positive integer n, the inequality $\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!} \le 2 - \dfrac{1}{n}$ is valid.

5. Let $a_1 = 1$ and $a_{n+1} = 3 - (1/a_n)$ for each integer $n \ge 1$. Prove that $1 \le a_n \le 3$ for all $n \in \mathbb{Z}^+$.

6. Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ for each positive integer n.

7. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for each positive integer n.

8. Prove that $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ for each positive integer n.

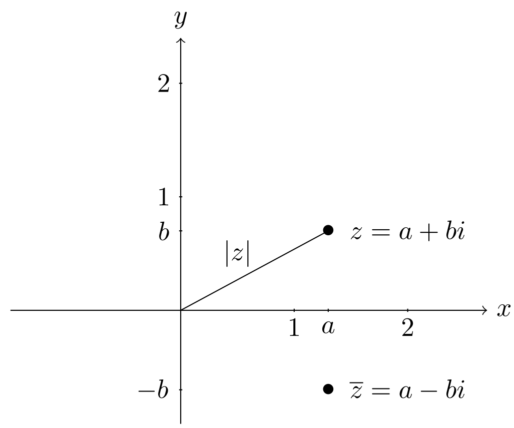
9. Let $\alpha = \big(1+\sqrt{5}\big)/2$ and $\beta = \big(1-\sqrt{5}\big)/2$; the numbers α and β are the solutions to the equation $x^2 = x + 1$. Prove that $f_n = \big(\alpha^n - \beta^n\big)/\sqrt{5}$ and $\ell_n = \alpha^n + \beta^n$ for each positive integer n.

We are going to work with some simple properties of complex numbers. A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. Although the number i seems strange (the i stands for imaginary), we will treat it like it appears, that is, we use $i^2 = -1$. For instance, we have

$$(1 + 2i)(3 - 5i) = 3 - 5i + 6i - 10i^2 = 13 + i.$$

We often use letters such as z and w to denote complex numbers and we use $\mathbb{C}$ to represent the set of all complex numbers. Note that $\mathbb{R} \subseteq \mathbb{C}$ by choosing $b = 0$. For a given a complex number $z = a + bi$, we let $\overline{z} = a - bi$ (this is called the conjugate of z) and $|z| = \sqrt{a^2 + b^2}$. Note that $|z|^2 = z\,\overline{z}$.

Since a and b are real numbers, the complex number $a + bi$ is related to the ordered pair (a, b) of real numbers. We are thus able to represent complex numbers graphically using the plane $\mathbb{R}^2$ as shown below. The x-axis is called the real axis and the y-axis is called the imaginary axis; we thus have a geometric picture for the complex plane $\mathbb{C}$.



Be sure that you see the geometric connections between z, $\overline{z}$, and $|z|$.

1. For complex numbers w and z, prove that $\overline{wz} = \overline{w}\,\overline{z}$.
2. Write the numbers $(2 - 3i)(5 - 4i)$ and $\dfrac{4 + 7i}{1 + i}$ in $a + bi$ form.

A well-known theorem involving complex numbers is the Fundamental Theorem of Algebra. It states that every polynomial of degree n has n roots in the complex plane.

3. Find the roots of the polynomial $P(z) = z^2 + 4z + 9$.
4. Find the roots of the polynomial $Q(z) = z^3 + 3z^2 + 5z + 3$.

Using Maclaurin series or properties of differential equations, it can be shown that $e^{ix} = \cos x + i \sin x$. It then follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

5. What are the values of $e^{i\pi}$, $e^{i\pi/2}$, and $e^{i2\pi}$?
6. Find seven different complex numbers z that satisfy $z^7 = -1$.
7. Prove that $2\cos(\pi/7)$ is a root of the polynomial $z^3 - z^2 - 2z + 1$.

Theorem: The equation $\displaystyle\sum_{k=1}^{n-1} \tan^2(k\pi/n) = n(n-1)$ is valid for each odd integer $n > 1$.

Proof: Let $n > 1$ be an odd integer. We first note that

$$\tan(k\pi/n) = \frac{1}{i}\Big(\frac{e^{ik\pi/n} - e^{-ik\pi/n}}{e^{ik\pi/n} + e^{-ik\pi/n}}\Big) = \frac{1}{i}\Big(1 - \frac{2}{e^{i2k\pi/n}+1}\Big)$$

and thus

$$\tan^2(k\pi/n) = -1 + \frac{4}{e^{i2k\pi/n}+1} - \frac{4}{\big(e^{i2k\pi/n}+1\big)^2}$$

for all $1 \le k \le n$. For these values of k, let $x_k = \big(e^{i2k\pi/n}+1\big)^{-1}$. Since (recall that n is odd)

$$x_k = \frac{1}{e^{i2k\pi/n}+1} \qquad \Leftrightarrow \qquad \Big(\frac{1}{x_k} - 1\Big)^n = 1 \qquad \Leftrightarrow \qquad (x_k - 1)^n = -x_k^n,$$

we see that the set $\{x_k : 1 \le k \le n\}$ contains the n distinct roots of the polynomial

$$\frac{(x-1)^n + x^n}{2} = \frac{1}{2}\left(x^n + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(-1)^k\right) = x^n - \frac{n}{2}\,x^{n-1} + \frac{n(n-1)}{4}\,x^{n-2} - \cdots + \frac{1}{2}\,x - \frac{1}{2}.$$

It then follows that

$$\sum_{k=1}^{n} x_k = \frac{n}{2}; \qquad \sum_{k=1}^{n} x_k^2 = \Big(\frac{n}{2}\Big)^2 - 2\cdot\frac{n(n-1)}{4} = \frac{n}{2} - \frac{n^2}{4};$$

and thus

$$\sum_{k=1}^{n-1} x_k = \frac{n-1}{2}; \qquad -\sum_{k=1}^{n-1} x_k^2 = \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4} = \Big(\frac{n-1}{2}\Big)^2.$$

Putting these results together, we obtain

$$\begin{aligned}\sum_{k=1}^{n-1}\tan^2(k\pi/n) &= \sum_{k=1}^{n-1}\big(-1 + 4x_k - 4x_k^2\big) = -\sum_{k=1}^{n-1} 1 + 4\sum_{k=1}^{n-1} x_k - 4\sum_{k=1}^{n-1} x_k^2 \\ &= -(n-1) + 2(n-1) + (n-1)^2 = n(n-1),\end{aligned}$$

as desired. The polynomial needed for this proof is quite simple. ∎

8. Use the properties of the numbers x_k and simple trigonometric identities to prove that

$$\sum_{k=1}^{n-1}\tan(k\pi/n) = 0 \qquad \text{and} \qquad \sum_{k=1}^{n-1}\sec^2(k\pi/n) = n^2 - 1$$

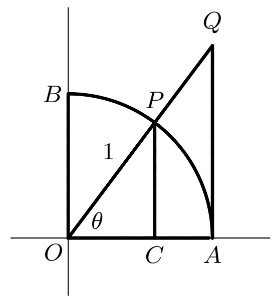
for all odd positive integers $n \ge 3$.

To find the derivative of the sine function, it is necessary to return to the definition of the derivative and determine some way of computing the limit of the difference quotient. In this case, some properties of the trigonometric functions and a few trigonometric identities provide the relevant information. The definition of the derivative yields

$$\begin{aligned}\frac{d}{dx}\sin x = \lim_{\theta\to 0}\frac{\sin(x+\theta)-\sin x}{\theta} &= \lim_{\theta\to 0}\frac{\sin x\cos\theta+\sin\theta\cos x-\sin x}{\theta}\\ &= \lim_{\theta\to 0}\Big(\cos x\,\frac{\sin\theta}{\theta}-\sin x\,\frac{1-\cos\theta}{\theta}\Big).\end{aligned}$$

To determine the limits of the quotients $\sin\theta/\theta$ and $(1-\cos\theta)/\theta$ as $\theta\to 0$, assume that θ is given in radians and consider the portion of the unit circle that lies in the first quadrant:

$A\ (1,0)$
$B\ (0,1)$
$C\ (\cos\theta, 0)$
$P\ (\cos\theta, \sin\theta)$
$Q\ (1, \tan\theta)$



area of triangle OCP is $\frac{1}{2}\sin\theta\cos\theta$
area of sector OAP is $\frac{1}{2}\,\theta$
area of triangle OAQ is $\frac{1}{2}\tan\theta$

From the figure, it is clear that the area of triangle OCP is less than the area of sector OAP which in turn is less than the area of triangle OAQ. Determining these areas in terms of θ and rearranging gives

$$\sin\theta\cos\theta < \theta < \tan\theta \quad\Rightarrow\quad \cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Although the figure indicates that θ is positive, this equation is valid for any small nonzero value of θ because $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)/(-\theta)=\sin\theta/\theta$. Since $\lim\limits_{\theta\to 0}\cos\theta = 1$, the squeeze law gives

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=1 \quad\text{or equivalently}\quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Using this limit and some algebra yields

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}=\lim_{\theta\to 0}\left(\frac{\sin\theta}{1+\cos\theta}\cdot\frac{\sin\theta}{\theta}\right)=0\cdot 1=0.$$

Given the values of these limits, the derivative of the function $\sin x$ is $\cos x$. Graphing $y=\sin x$, then using the graph to sketch its derivative makes this result seem very plausible. Using the above information and other facts about derivatives, we find that

$$\frac{d}{dx}\sin x=\cos x \qquad \frac{d}{dx}\tan x=\sec^2 x \qquad \frac{d}{dx}\sec x=\sec x\tan x$$
$$\frac{d}{dx}\cos x=-\sin x \qquad \frac{d}{dx}\cot x=-\csc^2 x \qquad \frac{d}{dx}\csc x=-\csc x\cot x$$

1. Verify the derivative for $\cos x$ two different ways; one using the definition of the derivative and another using the chain rule.
2. Use simple facts to verify the derivative formulas for the other four trigonometric functions.

3. Find and simplify the derivative of $f(x) = \cos^4(x^2)$.

4. Find all values of x in $[0, 2\pi]$ for which $g'(x) = 0$ given that $g(x) = \dfrac{\cos x}{2 + \sin x}$.

5. Evaluate each of the following limits, where r is a nonzero real number.

   a) $\displaystyle\lim_{\theta\to 0} \frac{\sin 2\theta}{\sin r\theta}$ **b)** $\displaystyle\lim_{\theta\to 0} \frac{\tan r\theta}{\theta}$ **c)** $\displaystyle\lim_{\theta\to 0} \frac{\tan r\theta}{\sin 7\theta}$

6. Show graphically that there is a point on the graph of $y = \sec x$ for which the tangent line goes through the origin. If the x-coordinate of such a point is a, what equation must a satisfy?

The derivatives of the inverse trigonometric functions follow fairly easily from the derivatives of the trigonometric functions and some trigonometric identities. Assuming that $\arcsin x$ is differentiable (this fact does require proof, but we will not concern ourselves with it), its derivative can be found using an identity and the chain rule:

$$\sin(\arcsin x) = x \quad\Rightarrow\quad \frac{d}{dx}\sin(\arcsin x) = 1 \quad\Rightarrow\quad \cos(\arcsin x)\cdot\frac{d}{dx}\arcsin x = 1 \quad\Rightarrow\quad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

As a simple example, note that

$$\frac{d}{dx}\arcsin(x^2/2) = \frac{1}{\sqrt{1-\left(x^2/2\right)^2}}\cdot x = \frac{x}{\sqrt{(4-x^4)/4}} = \frac{2x}{\sqrt{4-x^4}}.$$

The derivatives of the six inverse trigonometric functions are given below.

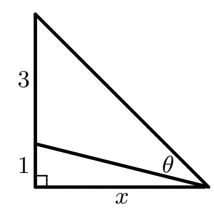
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

7. Verify the derivative formulas for the other five inverse trigonometric functions.

8. Find and simplify the derivative of $f(x) = x\arccos x - \sqrt{1-x^2}$.

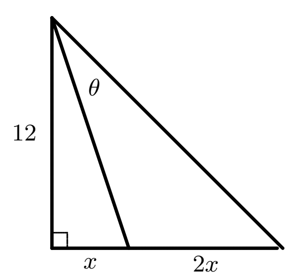
9. Find the value of x that will maximize the angle θ.



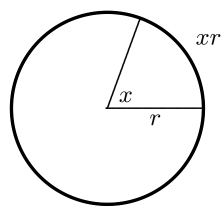
10. Find the maximum and minimum outputs of the function $h(x) = \sin^2 x + \cos x$ on the interval $[0, \pi]$.

11. A line of length 60 is split into equal thirds. The right and left thirds are then each bent upward through the same angle θ to form a (topless) trapezoid. Find the value of θ that will maximize the area of the trapezoid.

12. Find the value of $x \in (0, \infty)$ that will maximize the angle θ.



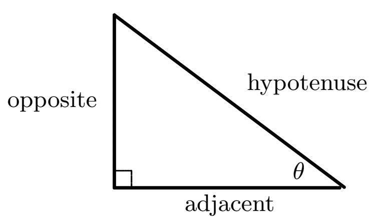
We assume that the reader is familiar with angles and angle measurement, both in degrees and radians. As a quick reminder, if x is a number between 0 and 2π, then the angle x radians is the angle cut off in a circle of radius r by an arc of length xr (see the figure).



$$360^\circ = 2\pi \text{ radians}$$
$$1^\circ = \frac{\pi}{180} \text{ radians}$$
$$\frac{180}{\pi}^\circ = 1 \text{ radian}$$

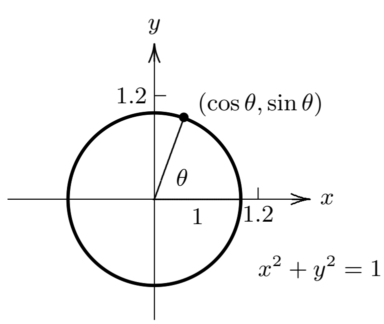
If $x > 2\pi$, then the angle is determined by "taking laps" in a counterclockwise direction. If $x < 0$, then the angle is determined by going in a clockwise direction.

The word "trigonometry" refers to the measurement of triangles. For acute angles, the trigonometric functions can be defined using the sides of a right triangle as in the figure below.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

However, in calculus, the trigonometric functions need to be defined for all real numbers. Given a real number θ, interpret θ as the radian measure of an angle with vertex at the origin and initial side the positive x-axis. The terminal side of this angle intersects the unit circle in a unique point. The x-coordinate of this point is defined to be $\cos\theta$ and the y-coordinate is defined to be $\sin\theta$. The other trigonometric functions are then defined in terms of $\sin\theta$ and $\cos\theta$.



$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

A number of relationships are clear from the definitions of the trigonometric functions. These include the fact that the trigonometric functions repeat every 2π units as well as the following identities:

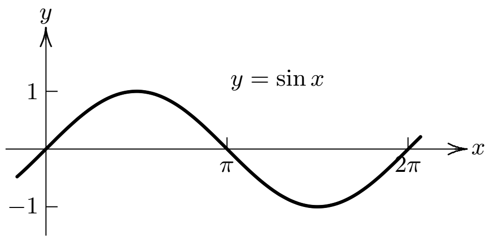
$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin(-\theta) = -\sin\theta \qquad \sin(\pi - \theta) = \sin\theta \qquad \sin(\pi + \theta) = -\sin\theta$$
$$\tan^2\theta + 1 = \sec^2\theta \qquad \cos(-\theta) = \cos\theta \qquad \cos(\pi - \theta) = -\cos\theta \qquad \cos(\pi + \theta) = -\cos\theta$$
$$1 + \cot^2\theta = \csc^2\theta \qquad \tan(-\theta) = -\tan\theta \qquad \tan(\pi - \theta) = -\tan\theta \qquad \tan(\pi + \theta) = \tan\theta$$

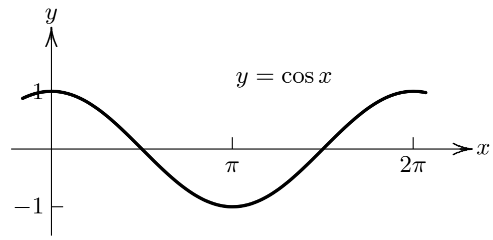
Another set of useful identities that follow from the symmetry of the circle are

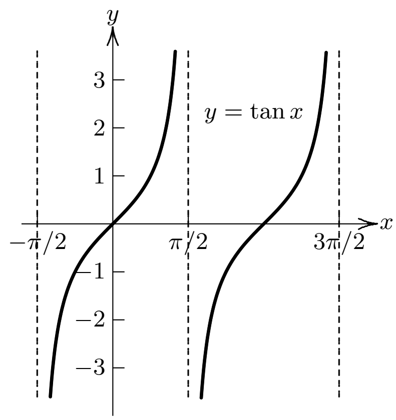
$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right); \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right); \qquad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right).$$

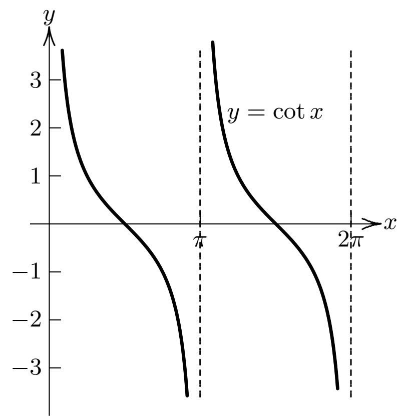
The prefix "co" in front of three of the trigonometric functions refers to the complement of an angle; for instance, the cosine of x is the sine of the complement of x.

In calculus, the angle will most often be denoted by x, where it is assumed that x is in radians. The graphs of the functions $\sin x$, $\cos x$, $\tan x$, and $\cot x$ are given below. Since the graphs are periodic (that is, they repeat every 2π units) only a portion of each graph is given.









1. Sketch (using your own knowledge) the graphs for $\sec x$ and $\csc x$.

The exact values of the trigonometric functions can be determined easily for some angles. These values are recorded in the following table and should be used when they appear in problems.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	$*$

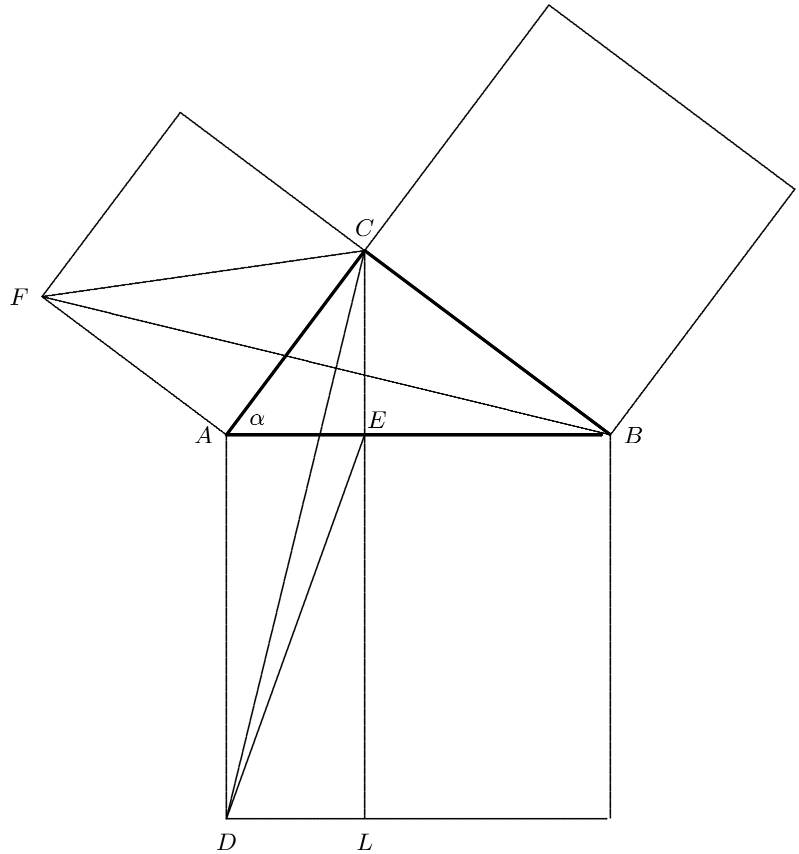
2. Use simple properties of right triangles to verify the above values.
3. Find the exact value of all the trigonometric functions for the angles (in radians) $2\pi/3$ and $7\pi/6$.

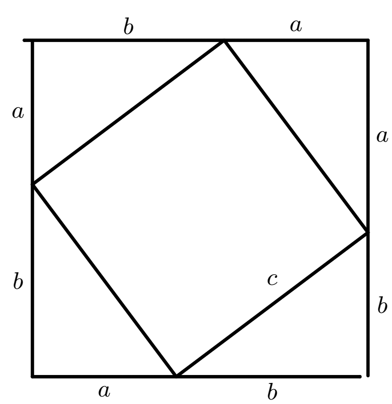
There are many other identities satisfied by the trigonometric functions. Some of these are listed below. If possible, you should commit these identities to memory. At the very least, it is important to know that such formulas exist and to be able to use them when necessary.

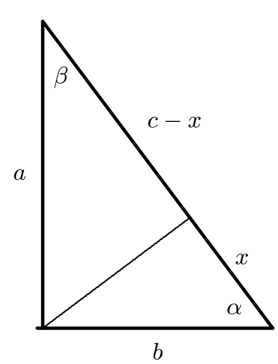
$$\begin{array}{lll}
\sin(x+y) = \sin x\cos y + \sin y\cos x & \sin 2x = 2\sin x\cos x & \sin^2 x = \dfrac{1-\cos 2x}{2} \\
\cos(x+y) = \cos x\cos y - \sin x\sin y & \cos 2x = \cos^2 x - \sin^2 x & \cos^2 x = \dfrac{1+\cos 2x}{2} \\
\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y} & \tan 2x = \dfrac{2\tan x}{1-\tan^2 x} & \tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}
\end{array}$$

4. Verify the identities listed above; the law of cosines mentioned later is needed for the second one.

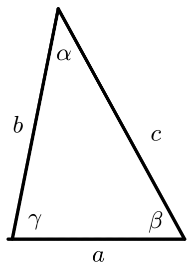
5. Use the figures below to prove the Pythagorean Theorem three different ways.







For triangles that do not have a right angle, the following relationships between the sides and angles of a triangle are sometimes useful.



law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

6. Prove the law of cosines and the law of sines.

Finally, we present the definitions for the six inverse trigonometric functions. The number $\arcsin(\frac{1}{2})$ represents the angle or arc (in radians) for which the value of the sine function is $\frac{1}{2}$. Since there are many angles for which this is true, we need to limit the range of potential answers in order to define a function. One way to proceed is the following. (Note the range of each function.)

i. For each real number $x \in [-1, 1]$, $\arcsin x$ is the unique real number taken from the interval $[-\pi/2, \pi/2]$ that satisfies $\sin(\arcsin x) = x$.
ii. For each real number $x \in [-1, 1]$, $\arccos x = \dfrac{\pi}{2} - \arcsin x$.
iii. For each real number x, $\arctan x = \arcsin\Big(\dfrac{x}{\sqrt{x^2+1}}\Big)$.
iv. For each real number x, $\operatorname{arccot} x = \dfrac{\pi}{2} - \arctan x$.
v. For each real number x that satisfies $|x| \geq 1$, $\operatorname{arccsc} x = \arcsin(1/x)$.
vi. For each real number x that satisfies $|x| \geq 1$, $\operatorname{arcsec} x = \dfrac{\pi}{2} - \operatorname{arccsc} x$.

7. Sketch the graphs for the six inverse trigonometric functions.

8. Simplify the expressions $\sec(\arctan(2x))$ and $\cos(2\arcsin x)$.

9. Find the exact value of all the trigonometric functions given that $\sin x = 2/3$ and $0 < x < \pi/2$.

10. Find the exact value of all the trigonometric functions given that $\tan x = -4$ and $\pi/2 < x < \pi$.

11. Find all of the values of x in the interval $[-2\pi, 4\pi]$ that satisfy $\cos x = 1/2$.

12. Find three solutions to the equation $1 + \tan x = 0$.

13. Without a calculator, find the exact value of each of the following.

a) $\arcsin(1/\sqrt{2}\,)$ **b)** $\arcsin(-1/2)$ **c)** $\arccos(-1/2)$
d) $\arctan(1/\sqrt{3})$ **e)** $\operatorname{arcsec}(-\sqrt{2})$ **f)** $\operatorname{arccsc}(-2/\sqrt{3})$

14. Simplify each of the following expressions. Indicate the values of x for which each is defined.

a) $\tan(\arcsin x)$ **b)** $\sin(\arctan x)$ **c)** $\cos(2\arcsin x)$

Students will spend about 30 minutes working alone (and without access to notes or electronic devices) on the problems below. You can work on whichever problems strike your interest and/or require knowledge that you remember from previous math classes. We will then discuss these problems as a class and consider various strategies for solving them.

1. Find the distance from the line $4x + 3y = 24$ to the origin. Also, find the distance from this line to the point $(10, 3)$.
2. There are two points on the circle $x^2 + y^2 = 1$ for which the tangent line passes through the point $(7, 1)$. Find the (x, y) coordinates of both these points.
3. Find the minimum distance from a point on the parabola $y = x^2$ to the point $(0, 2)$.
4. Find the ordered pair (s, t) that satisfies the equation $x^2 - xy + y^2 = 1$ and has the largest possible value for t.
5. Evaluate $\displaystyle\lim_{n\to\infty} \frac{(n+1)+(n+2)+(n+3)+\cdots+2n}{n^2}$.
6. Evaluate the following three limits:

$$\lim_{x\to 0^+} \frac{\ln(1-x)\sin x}{1-\cos^2 x}, \qquad \lim_{x\to 0^+} \frac{\ln(1-x)-\sin x}{1-\cos^2 x}, \qquad \lim_{x\to 0^+} \frac{\ln(1+x)-\sin x}{1-\cos^2 x}.$$

   One of these limits appears in the movie Mean Girls.

7. Let R be the region under the curve $y = 4/x$ and above the x-axis on the interval $[1, 4]$. Find (a) a vertical line and (b) a horizontal line that divides the region R into two pieces of equal area.