Math 456 Assignments for Spring 2024 (due on given date)

for Tuesday, January 16

1. We will spend most of our time working through Chapters 7 and 8 in the text (R. Gordon, *Real Analysis, A First Course*, 2nd ed.). We will treat this as a Tu/Th class in terms of due dates for assignments and we can meet when necessary to discuss the material. Homework will be assigned every class period. These assignments will include a fair amount of reading and some problems; a few of these will be specified to be written up carefully and submitted for grading. (It would be preferable for you to print your solutions and slide them under my office door, but if that is a hassle, we can work something out.) For these assignments, you can contact me if you need some help. There will be several assignments during the semester, sometimes assigned a week in advance, for which you will be required to work alone, that is, no help from other books or the Internet.

   There will be two exams during the semester (the week beginning Feb. 26 and the week beginning Apr. 22, with timing at your discretion) and a comprehensive final exam (scheduled at a suitable time for you during finals week). Each of the hour exams represents 20% of your course grade and the final exam represents another 25% of your course grade. The regular graded homework counts as 20% and the special homework (the problems you must do on your own) as 15% of your course grade. I will make an attempt to keep you posted on your grade for the course as the semester progresses.

for Thursday, January 18

1. Flip through Chapter 6 of the textbook, noting all of the theorems and working through some of the proofs. Read the proofs of Theorems 6.9, 6.12, and 6.18 carefully.
2. Work through Exercises 7, 8, 10, and 17 (you may use Exercise 16) in Section 6.4.
3. Turn in a solution for Exercise 6.4.10 for the special case $p = 3$ and $q = 2$. You may refer to Exercise 7 without giving a proof of that exercise.

for Tuesday, January 23

1. Read Section 7.1. Study the examples very carefully and do your best to understand this new concept.
2. Look over all of the exercises in Section 7.1 to get a sense for these problems then do some of the ones that look interesting to you.
3. Turn in solutions for Exercises 5 and 14 in Section 7.1.

for Thursday, January 25

1. Read Section 7.2. Uniform convergence is a very important topic in real analysis.
2. Look over all of the exercises in Section 7.2 to get a sense for these problems then do some of the ones that look interesting to you.
3. Turn in solutions for exercises 4c and 5 in Section 7.2.

for Tuesday, January 30

1. Read Section 7.3.
2. Do exercises 1, 2, 3, 5, 9, 13, 14, and 18 in Section 7.3.
3. Turn in solutions for exercises 5b and 5c in Section 7.3. For part (b), the hypothesis (in case it is not clear) is that all of the functions f_n are continuous and that the sequence $\{f_n\}$ is equicontinuous. For part (c), assume that $I = [a, b]$.
4. There is a special assignment due next Tuesday; see that date for the assignment.

for Thursday, February 1

1. Read Section 7.4. Hopefully, much of this material is familiar to you from calculus.
2. Do exercises 1, 2, 3, 4, 5, 12, 14, 15, and 16 in Section 7.4.
3. Turn in solutions for exercises 2c and 20 in Section 7.4; see the bottom of page 263 for a suggestion for 20.

for Tuesday, February 6

1. Read Section 7.5, keeping track of any questions that arise.
2. Do exercise 7 in Section 7.5.
3. A special assignment is due by 5:00 pm on this date. The three problems are listed below:

i. Exercise 7.3.6. An outline for this proof is there to guide you, but you should write out the proof without reference to the various steps.

ii. Exercise 7.4.19. In addition, represent the function f as a rational function (that is, express the sum of the power series as a more familiar function).

iii. Exercise 7.7.12.

for Thursday, February 8

1. Read Section 7.5 again, noting any questions that arise.
2. Do exercises 1, 12, 14, and 18 in Section 7.5.
3. Turn in solutions for exercises 1 and 14 in Section 7.5.

for Tuesday, February 13

1. Read Section 7.6 through the statement of Theorem 7.29.
2. Do exercises 1, 3, 4, 5a, 6, and 8 in Section 7.6.
3. Turn in solutions for Exercises 1b and 3 in Section 7.6.

for Thursday, February 15

1. Finish reading Section 7.6. Do your best to make sense of the proofs for these two important results.
2. Do exercises 19, 20, 21, and 22 in Section 7.6.
3. Turn in solutions for exercises 20b and 22 in Section 7.6. For exercise 22, you may use the result of Exercise 5.3.10.
4. There is a special assignment due next Tuesday; see that date for the assignment.

for Tuesday, February 20

1. Look over Chapter 7 to review the key ideas and results.
2. Do exercises 1, 7, 11, 27, 28, and 33 in Section 7.7. If the ideas in Exercise 7.7.6 intrigue you, spend some time pondering this problem.
3. The second special assignment is due this day. The three problems are listed below:

i. Suppose that the sequence $\{f_n\}$ converges uniformly to a continuous function f on $[a, b]$ and that the equation $f_n(x) = 0$ has exactly one solution in $[a, b]$ for each positive integer n. Prove that the equation $f(x) = 0$ has a solution in $[a, b]$.

ii. Prove that the Cauchy product of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ with itself converges. Letting h_k represent the kth partial sum of the harmonic series, recall that $h_k = \gamma_k + \ln k$ and $\{\gamma_k\}$ converges (see page 177).

iii. Exercise 7.6.24, parts (a) and (b) only.

for Thursday, February 22

0. Technically, there is no class today due to the Power and Privilege Symposium. If this creates any difficulty, please let me know.
1. Read the introduction to Chapter 8 and the portion of Section 8.1 through the proof of Theorem 8.5.
2. Do exercises 1 through 10, excluding 7, in Section 8.1.
3. Turn in solutions for exercises 6 and 8 in Section 8.1. These two exercises are intended to have short and simple solutions.

for Tuesday, February 27

1. Finish reading Section 8.1.
2. Work on exercises 11 through 36 in Section 8.1, making note of which exercises you find challenging so that we can discuss them at some point in time.
4. Turn in solutions for exercises 27 and 28 in Section 8.1.

for Thursday, February 29

1. Read Section 8.2 through the proof of Theorem 8.14.
2. Work on exercises 1–11 in Section 8.2, making note of which exercises you find challenging.
3. The third special assignment is due this day. The three problems for this assignment are exercises 38, 39, and 40b in Section 8.1.
4. We need to schedule a 90 minute exam for next week. It will cover all of the parts of Chapter 7 that have been assigned. You should know the important theorems and concepts and how to work with them, and you should know the Maclaurin series for e^x, $\sin x$, and $\cos x$. I realize that this is vague, but at this stage of your mathematics career, you should be able to determine what theorems are important and understand the key concepts that have been covered.

for Tuesday, March 5

1. We have an exam on this day, covering the material from Chapter 7.

for Thursday, March 7

1. Finish reading Section 8.2 and then read Section 8.3 through the proof of Theorem 8.22.
2. Work on exercises 16, 17, 20, and 22 in Section 8.2 and exercises 4, 5, 6, and 13 in Section 8.3.
3. Turn in solutions for exercise 16 in Section 8.2 and exercise 13 in Section 8.3.

for Tuesday, March 26

1. Finish reading Section 8.3.
2. Work on exercises 21, 23, and 33 in Section 8.3.
3. Turn in a solution for exercise 21 in Section 8.3.

for Thursday, March 28

1. Read Section 8.4 through the proof of Theorem 8.34.
2. Work on exercises 2, 5, 8, 9, 10, 19, 20, and 21 in Section 8.4.
3. Turn in solutions for exercises 22 and 23 in Section 8.4. (You may use previous parts of Theorem 8.34 in these proofs.)

for Tuesday, April 2

1. Finish reading Section 8.4.
2. Work on exercises 32, 33, and 34 in Section 8.4.
3. Our fourth special assignment is due on this day. Turn in solutions for Exercise 8.3.34, Exercise 8.4.18ab (use the Lindelöf covering theorem in your proof of part a), and Exercise 8.4.27.

for Thursday, April 4

1. Read Section 8.5 through the statement of Theorem 8.46. Convince yourself that the triangle inequality holds for each of the listed metric spaces.
2. Work on exercises 1–16 in Section 8.5, making note of which exercises seem challenging. For part (e) of Exercise 10, use $[0, \pi]$ for the interval rather than a generic $[a, b]$. In addition, consider the three functions t, $1 - t$, and t^2 on the interval $[0, 1]$.
3. Turn in solutions for exercises 14 and 15 in Section 8.5.

for Tuesday, April 9

1. Read Section 8.5 through the statement of Theorem 8.53.
2. Work on exercises 17–36 in Section 8.5. Give each of them some thought, solve some of them completely, and keep track of any questions or difficulties that arise. It is not necessary to solve all of these exercises.
3. Turn in solutions for exercises 29 and 36 in Section 8.5.
4. Write up solutions for exercises 18e (there are several parts to this exercise) and 21ab (with some thought, the integrals are easy to evaluate). These two exercises are comprise our sixth special assignment; it is due this coming Thursday.

for Thursday, April 11

1. Work on exercises 37–43 in Section 8.5. Keep track of any questions or difficulties that arise.
2. Turn in the special assignment mentioned in the Tuesday (4/9) assignment.

for Tuesday, April 16

1. Read Section 8.5 through Theorem 8.61.
2. Work on exercises 44–58.
2. Turn in solutions for exercises 53 and 58b.

for Thursday, April 18

1. Carefully read all parts of Exercise 63 and think a little about how you would solve these problems. Also, work on exercises 79 and 82.
2. Turn in solutions for exercises 63b and 79.
3. We need to schedule an exam for early next week so think about which day might work best for you.

for Tuesday, April 23

1. Look over the parts of Section 8.5 we have covered thus far and record any questions you have.
2. Turn in a solution for exercise 77.

for Thursday, April 25

1. We have our second exam, covering Sections 8.2 through the part of Section 8.5 we have covered thus far. Flip through these sections, focusing on the main definitions and results. As with the first exam, the exam will be for 90 minutes.

for Tuesday, April 30

1. Read the remaining portion of Section 8.5 and look through all of the remaining exercises. Identify those concepts, theorems, and exercises that intrigue you so that we can focus on those for the limited time remaining. Send me an email on which group of exercises you might want to work through.
2. Work on exercises 86 and 87 as well as any exercises that you found intriguing as well as the parts of the test that you did not finish.
3. Turn in a solution for Exercise 86.

for Thursday, May 2

1. Work on exercise 91.
2. Turn in solutions for parts (a) and (g) of Exercise 91.

for Tuesday, May 7

1. Turn in a solution for Exercise 103.

Exercise 8.4.36: Let $\{O_n\}$ be a sequence of open dense sets. Use the previous exercise to prove that $\bigcap_{n=1}^{\infty} O_n$ is a dense set.

Solution: Let I be any interval. By the previous exercise, it is sufficient to show that I contains a point of $\bigcap_{n=1}^{\infty} O_n$. Without loss of generality, we may assume that $I = [a, b]$. Since O_1 is a dense set, the interval (a, b) contains a point of O_1. Since O_1 is open, there exists an interval $[a_1, b_1]$ such that $[a_1, b_1] \subseteq [a, b]$ and $[a_1, b_1] \subseteq O_1$. Similarly, there exists an interval $[a_2, b_2]$ such that $[a_2, b_2] \subseteq [a_1, b_1]$ and $[a_2, b_2] \subseteq O_2$. Continuing this process, we obtain a nested sequence $\{[a_n, b_n]\}$ of intervals such that $[a_n, b_n] \subseteq O_n$ for each n. Let z be a point that lies in all of the intervals and note that $z \in [a, b]$. For each positive integer n, we find that $z \in [a_n, b_n] \subseteq O_n$. This shows that z belongs to the set $\bigcap_{n=1}^{\infty} O_n$. ■

Exercise 8.4.38: Let E be a set that can be expressed as $E = \bigcup_{n=1}^{\infty} E_n$, where each E_n is nonempty, closed, and nowhere dense. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \notin E; \\ 1/p, & \text{if } x \in E, \text{ where } p = \min\{n : x \in E_n\}. \end{cases}$$

Prove that the set of points at which f is continuous is $\mathbb{R} \setminus E$.

Solution: Suppose that $x \in E$ and let $f(x) = 1/p$. By the Baire Category Theorem, each interval of the form $[x - \delta, x + \delta]$ contains a point $t \in \mathbb{R} \setminus E$. It follows that $|f(t) - f(x)| = 1/p$ and thus f is not continuous at x. Now suppose that $x \notin E$. Let $\epsilon > 0$ and choose a positive integer N such that $1/N < \epsilon$. Since the set $A = \bigcup_{n=1}^{N} E_n$ is closed, the set $\mathbb{R} \setminus A$ is an open set that contains x. It follows that there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq \mathbb{R} \setminus A$. For each real number t that satisfies $|t - x| < \delta$, we find that either $t \in \mathbb{R} \setminus E$ or $t \in E$ with $\min\{n : x \in E_n\} > N$. It follows that $|f(t) - f(x)| = f(t) < 1/N < \epsilon$ and we conclude that the function f is continuous at x. Hence, the set of points at which f is continuous is $\mathbb{R} \setminus E$. ■

Exercise 8.4.39: Let E be a nonempty, bounded, perfect, nowhere dense set, let $a = \inf E$, let $b = \sup E$, and let $[a, b] \setminus E = \bigcup_{k=1}^{\infty} (a_k, b_k)$, where the intervals (a_k, b_k) are disjoint. Let $\{y_k\}$ be a sequence of real numbers, and for each positive integer k, let c_k be the midpoint of $[a_k, b_k]$. Define a function $f\colon [a, b] \to \mathbb{R}$ by $f(x) = 0$ for $x \in E$, $f(c_k) = y_k$ for each k, and letting f be linear on each of the intervals $[a_k, c_k]$ and $[c_k, b_k]$.

(a) Prove that f has the intermediate value property.

(b) Prove that there exists a sequence $\{f_n\}$ of continuous functions that converges pointwise to f on $[a, b]$.

(c) Prove that f is continuous on $[a, b]$ if and only if $\{y_k\}$ converges to 0.

(d) Suppose that $|y_k| > m$ for all k for some positive constant m. Find the set of points of continuity of f.

Solution: It is intuitively clear that f is a Darboux function, but the details are somewhat tedious. Suppose that $a \le c < d \le b$ and that v is a number between $f(c)$ and $f(d)$, assuming that $f(c) \ne f(d)$. We know that at least one of c or d does not belong to E. Suppose first that $c \in E$ and $d \in (a_k, b_k)$ for some positive integer k. Then v is between $0 = f(a_k) = f(c)$ and $f(d)$. Since f is continuous on $[a_k, b_k]$, there exists a number $u \in (a_k, d) \subseteq (c, d)$ such that $f(u) = v$. The proof for the case in which $d \in E$ and $c \in (a_k, b_k)$ for some positive integer k is similar. Finally, suppose that $c \in (a_j, b_j)$ and $d \in (a_k, b_k)$ for some positive integers j and k. The result is trivial if $j = k$ so we assume that $j \ne k$. Since E is a perfect set, we see that $c < b_j < a_k < d$, so if $v = 0$, then $a_k \in (c, d)$ and $f(a_k) = v$. If $v \ne 0$, consider the function $g\colon [a, b] \to \mathbb{R}$ defined by $g(x) = f(x)$ for $x \in [a_j, b_j] \cup [a_k, b_k]$ and $g(x) = 0$ otherwise. Since g is continuous on $[a, b]$, there exists a point $u \in (c, d)$ such that $g(u) = v$. Since $v \ne 0$, we must have $u \in (a_j, b_j) \cup (a_k, b_k)$ and thus $f(u) = g(u) = v$. It follows that f has the intermediate value property on $[a, b]$.

Let $I_k = (a_k, b_k)$ and for each positive integer n, let $f_n = \sum_{k=1}^{n} f\chi_{I_k}$. (Recall that χ_A is the function that equals 1 when the input belongs to A and 0 otherwise.) Each of these functions is continuous on $[a, b]$ (a finite sum of continuous functions) and the sequence $\{f_n\}$ converges pointwise to f on $[a, b]$. This proves part (b). Note that the Weierstrass M-test gives uniform convergence if the series $\sum_{k=1}^{\infty} |y_k|$ converges. However, as part(c) indicates, the convergence can be uniform even when this series diverges. For the record, part (b) shows that f is a Baire class one function.

We now turn to part (c). Suppose that f is continuous on $[a, b]$ and let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in [a, b]$. Since the series $\sum_{k=1}^{\infty} (b_k - a_k)$ converges (its sum is at most $b - a$), there exists a positive integer K such that $b_k - a_k < \delta$ for all $k \ge K$. For each $k \ge K$, it is clear that $|c_k - a_k| < \delta$ and thus $|y_k| = |f(c_k) - f(a_k)| < \epsilon$. This shows that the sequence $\{y_k\}$ converges to 0.

Now suppose that $\{y_k\}$ converges to 0. We will show that the sequence $\{f_n\}$ defined in the solution for part (b) converges uniformly to f on $[a, b]$. Let $\epsilon > 0$. Since $\{y_k\}$ converges to 0, there exists a positive integer N such that $|y_k| < \epsilon$ for all $k \geq N$. Fix $n \geq N$ and compute

$$|f(x) - f_n(x)| = \begin{cases} 0, & \text{for } x \in E \cup \bigcup_{k=1}^{n} I_k; \\ |f(x)| \leq |y_j|, & \text{for } x \in I_j,\ j > n. \end{cases}$$

This shows that $|f(x) - f_n(x)| < \epsilon$ for all $x \in [a, b]$. Therefore, the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$. It follows that the function f is continuous on $[a, b]$. Rather than this more or less direct proof of uniform convergence, we can use Theorem 7.4. Noting that

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [a, b]\} = \sup\{|y_j| : j > n\},$$

we find that the sequence $\{M_n\}$ converges to 0.

Finally, suppose that $|y_k| > m$ for all k for some positive constant m. It is clear that f is continuous at each point of $[a, b] \setminus E$ since it is linear on each interval forming the complement of E. Suppose that $x \in E$ and let $\delta > 0$. Since E is a perfect set, there exists a point $y \in E$ such that $0 < |y - x| < \delta$. Choose an index k so that the interval (a_k, b_k) is a subset of the interval with endpoints x and y; this is possible since the set E is nowhere dense. Then $|c_k - x| < \delta$ and $|f(c_k) - f(x)| = |y_k| > m$. It follows that f is not continuous at x. Hence, the set of points at which f is continuous is the set $[a, b] \setminus E$. ■