Exercise 4.7.10: Let B be the collection of all sequences of 0's and 1's for which the number of 1's is finite. In other words,

$$B = \{\{b_i\} : b_i \in \{0,1\} \text{ for all } i \text{ and there exists } n \in \mathbb{Z}^+ \text{ such that } b_i = 0 \text{ for all } i \geq n\}.$$

One example of an element of the set B is the sequence

$$1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,\ldots.$$

Prove that the set B is countably infinite.

Solution: For each nonnegative integer n, let $B_n = \{\{b_i\} \in B : b_i = 0 \text{ for all } i > n\}$. Since the set B_n contains 2^n elements, each B_n is a finite set and thus countable. Furthermore, by the definition of the set B, we see that $B = \bigcup_{n=0}^{\infty} B_n$. It follows that B is a countable set. Since the set B is clearly infinite (the 'standard' basis $\{\mathbf{e}_n : n \in \mathbb{Z}^+\}$ would give an infinite subset of B), we conclude that B is countably infinite.

Consider the function $f\colon B \to \mathbb{Z}^+ \cup \{0\}$ defined by $f(\{b_i\}) = \sum_{i=1}^{\infty} b_i 2^{i-1}$. (Given the definition of the set B, the sum is actually a finite sum.) To prove that f is injective, suppose $\{a_i\}$ and $\{b_i\}$ are two different elements of B. Choose a positive integer N such that $a_N \neq b_N$ and $a_i = b_i$ for all $i > N$. Without loss of generality, we may assume that $a_N = 1$ and $b_N = 0$. We then have

$$f(\{a_i\}) - f(\{b_i\}) = \sum_{i=1}^{\infty}(a_i - b_i)2^{i-1} = \sum_{i=1}^{N}(a_i - b_i)2^{i-1} \geq 2^{N-1} - \sum_{i=1}^{N-1} 2^{i-1} = 2^{N-1} - \left(2^{N-1} - 1\right) = 1.$$

This shows that $f(\{a_i\}) \neq f(\{b_i\})$ and we conclude that f is injective. The set $f(B)$ is countable since it is a subset of $\mathbb{Z}^+ \cup \{0\}$. Since B is in a one-to-one correspondence with $f(B)$, the set B is countably infinite.

If desired, we can use strong induction to show that f is a surjection. It is easy to verify that the integers $0, 1, 2, \ldots, 8$ are in the range of f. Suppose that all of the integers $0, 1, 2, \ldots, n$ are in the range of f for some positive integer $n \geq 8$. Choose a positive integer k such that $2^{k-1} < n+1 \leq 2^k$. If $n+1 = 2^k$, then $n+1$ is the image of the sequence composed of all 0's with a 1 in the $k+1$ position. Suppose that $2^{k-1} < n+1 < 2^k$ and consider the integer $m = n+1-2^{k-1}$. Since $0 \leq m \leq n$, the induction hypothesis tells us that m is in the range of f. Let $m = f(\{b_i\})$ and, since $m < 2^{k-1}$, note that $b_i = 0$ for all $i \geq k$. Consider the sequence $\{c_i\}$, where $c_i = b_i$ for all $i \neq k$ and $c_k = 1$. Then $\{c_i\}$ is in the set B and we have

$$f(\{c_i\}) = f(\{b_i\}) + 2^{k-1} = m + 2^{k-1} = n+1.$$

This shows that $n+1$ is in the range of f. By the Principle of Strong Induction, it follows that the range of f is $\mathbb{Z}^+$, that is, the function f is surjective.

Let $\{p_i\}$ be the sequence of primes and define a function $g\colon B \to \mathbb{Z}^+$ by $g(\{b_i\}) = \prod_{i=1}^{\infty} p_i^{b_i}$. (As above, the product only involves a finite number of terms that are greater than 1.) By the Fundamental Theorem of Arithmetic, it is clear that g is an injective function. (The function g in this case is certainly not surjective since no prime appears to a power larger than 1; the range of g is the collection of all square-free positive integers.) As above, it follows that B is countably infinite.

Exercise 4.8.2: Let A be the collection of all sequences of 0's and 1's. In other words,

$$A = \{\{a_i\} : a_i \in \{0, 1\} \text{ for all } i\}.$$

One simple example of an element of the set A is the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots.$$

a) Use Cantor's diagonal process to prove that the set A is uncountable.
b) Prove that the collection of all subsets of positive integers is uncountable by establishing a one-to-one correspondence between $\mathcal{P}(\mathbb{Z}^+)$ and the set A.
c) Explain and prove the statement $2^{\aleph_0} = c$. (Compare with Exercise 7 below.)

Solution: To prove part (a), suppose that $f\colon \mathbb{Z}^+ \to A$ is an injection. Denote $f(n)$ by $\{x_{n,i}\}$ for each $n \in \mathbb{Z}^+$. Consider the sequence $\{a_i\}$ defined by $a_i = 0$ if $x_{i,i} = 1$ and $a_i = 1$ if $x_{i,i} = 0$. Then $\{a_i\} \in A$. Since the sequences $\{x_{n,i}\}$ and $\{a_i\}$ have different numbers in the nth position, we see that $f(n) \neq \{a_i\}$ for all $n \in \mathbb{Z}^+$. This means that $\{a_i\}$ is not in the range of f. It follows that f is not surjective. Since every injection $f\colon \mathbb{Z}^+ \to A$ fails to be a surjection, there is no bijection between $\mathbb{Z}^+$ and A. Hence, the set A is uncountable.

Consider the function $f\colon A \to \mathcal{P}(\mathbb{Z}^+)$ defined by $f(\{a_i\}) = \{i \in \mathbb{Z}^+ : a_i = 1\}$. Suppose that $\{a_i\}$ and $\{b_i\}$ are two distinct elements of A. Since the two sequences are different, there exists an index n such that $a_n \neq b_n$. Without loss of generality, we may assume that $a_n = 1$ and $b_n = 0$. It follows that $n \in f(\{a_i\})$ but $n \notin f(\{b_i\})$. This shows that $f(\{a_i\}) \neq f(\{b_i\})$. Hence, the function f is injective. Now suppose that S is a set of positive integers. Define a sequence $\{c_i\}$ by $c_i = 1$ if $i \in S$ and $c_i = 0$ if $i \notin S$. It is then clear that $f(\{c_i\}) = S$. Hence, the function f is surjective. We have thus established a one-to-one correspondence between $\mathcal{P}(\mathbb{Z}^+)$ and the set A. Using the result in part (a), we find that the set $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Using binary expansions, we can establish a one-to-one correspondence between A and the interval $[0, 1]$. (Some care is required here since some real numbers have two binary expansions, one ending with all 0's and one ending with all 1's.) Hence, there is a one-to-one correspondence between $\mathcal{P}(\mathbb{Z}^+)$ and the interval $[0, 1]$. Since the cardinality of $\mathcal{P}(\mathbb{Z}^+)$ is $2^{\aleph_0}$ and the cardinality of $[0, 1]$ is c, we find that $2^{\aleph_0} = c$. ∎