Extra Notes for Section 1.12

Example 1: Find the maximum and minimum outputs for the function f defined by $f(x) = x^2 + \dfrac{32}{x}$ on the interval $[1, 4]$.

Solution: We first find the critical inputs of the function f that lie in the interval $[1, 4]$. Since

$$f'(x) = 2x - \frac{32}{x^2} = \frac{2(x^3 - 16)}{x^2},$$

the only critical input is $\sqrt[3]{16}$. Using this value, along with the endpoints, we compute the following outputs

$$\begin{aligned} f(1) &= 1 + 32 = 33; \\ f\left(\sqrt[3]{16}\,\right) &= 16^{2/3} + 2 \cdot 16^{2/3} = 3 \cdot 4^{4/3} = 12 \cdot 4^{1/3} \approx 19.0488; \\ f(4) &= 16 + 8 = 24. \end{aligned}$$

Hence, on the interval $[1, 4]$, the maximum output of f is 24 and the minimum output is $12\sqrt[3]{4}$. ■

Example 2: Suppose that x and y are positive numbers that are each greater than or equal to 1 and whose product is 20. Choose x and y so that the quantity $2x + 5y$ is as small as possible.

Solution: Since $xy = 20$, we know that $y = 20/x$. We thus want to minimize the outputs of the function f defined by

$$f(x) = 2x + \frac{100}{x}$$

on the interval $[1, 20]$. (The upper bound for x occurs when y is as small as possible, that is, when $y = 1$.) Using the fact that

$$f'(x) = 2 - \frac{100}{x^2} = \frac{2(x^2 - 50)}{x^2},$$

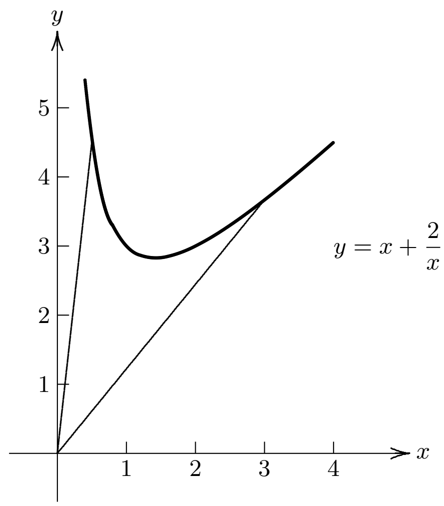
we find that the only relevant critical input for f is $x = 5\sqrt{2}$. Checking the outputs of f at this point and the endpoints yields

$$\begin{aligned} f(1) &= 2 + 100 = 102; \\ f\left(5\sqrt{2}\,\right) &= 10\sqrt{2} + \frac{20}{\sqrt{2}} = 20\sqrt{2} \approx 28.284; \\ f(20) &= 40 + 5 = 45. \end{aligned}$$

Therefore, the quantity $2x + 5y$ is as small as possible when $x = 5\sqrt{2}$ and $y = 2\sqrt{2}$. ■

Example 3: Consider the portion of the graph of $y = x + \dfrac{2}{x}$ that lies in the first quadrant. Find the minimum distance from a point on this graph to the origin.

Solution: Referring to the graph shown below, we are interested in finding the line of shortest length that connects the origin to a point on the graph. Lines connecting the origin to the points corresponding to $x = 1/2$ and $x = 3$ are drawn in the figure.



A typical point on the graph has coordinates $\left(x, x + \dfrac{2}{x}\right)$. Using the distance formula, the distance from this point to the origin is $\sqrt{x^2 + \left(x + \dfrac{2}{x}\right)^2}$. In order to minimize this quantity, it is sufficient to find the minimum output of the function S defined by

$$S(x) = x^2 + \left(x + \frac{2}{x}\right)^2.$$

This is an important idea to understand. For positive quantities q, the functions q and q^2 have their extreme outputs at the same inputs. The algebra is much simpler when the square root is eliminated. From the figure, we see that it is sufficient to find the minimum value of S on the interval $[\frac{1}{2}, 3]$. We first find the derivative of the function S:

$$S(x) = x^2 + x^2 + 4 + \frac{4}{x^2} \qquad \text{and} \qquad S'(x) = 4x - \frac{8}{x^3} = \frac{4(x^4 - 2)}{x^3}.$$

The relevant critical input is $x = \sqrt[4]{2}$. Since

$$S(1/2) = \frac{1}{2} + 4 + 16 = 20.5;$$
$$S\left(\sqrt[4]{2}\right) = 2\sqrt{2} + 4 + \frac{4}{\sqrt{2}} = 4\sqrt{2} + 4 \approx 9.6568;$$
$$S(3) = 18 + 4 + \frac{4}{9} \approx 22.444;$$

the minimum output of S occurs when $x = \sqrt[4]{2}$. Recalling that $S(x)$ represents the square of the distance, we find that the minimum distance from the curve to the origin is

$$2\sqrt{1 + \sqrt{2}} \approx 3.107548,$$

which occurs at the point $\left(\sqrt[4]{2}, \sqrt[4]{2} + \sqrt[4]{8}\right)$ on the graph. ■

Extra Notes for Section 1.13

Example 1: Consider the function f defined by $f(x) = x - \frac{4}{x^2}$. Determine the intervals on which f is increasing and those on which it is decreasing.

Solution: We begin by finding the derivative of the function f:

$$f'(x) = 1 + \frac{8}{x^3} = \frac{x^3+8}{x^3}.$$

The critical inputs for f are $x = -2$ (where $f'(x) = 0$) and $x = 0$ (where both $f(x)$ and $f'(x)$ are not defined). These two points give us three intervals to consider.

interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
sign of f'	positive	negative	positive
property of f	increasing	decreasing	increasing

Hence, the function f is increasing on the intervals $(-\infty, -2]$ and $(0, \infty)$, and decreasing on the interval $[-2, 0)$. Note that the endpoint -2 is included when relevant, while the endpoint 0 is excluded. This is due to the fact that f is not defined when $x = 0$. ■

Example 2: Let a be a positive constant and consider the function f defined by $g(x) = \sqrt{x} - ax$. Determine the intervals on which g is increasing and those on which it is decreasing.

Solution: We first note that the domain of g is the interval $[0, \infty)$. Using the fact that a is a positive constant, we obtain

$$g'(x) = \frac{1}{2\sqrt{x}} - a = \frac{1 - 2a\sqrt{x}}{2\sqrt{x}}.$$

There is a critical input at $x = 0$ but this is an endpoint of the domain of g. The only critical input that is positive is $x = 1/(4a^2)$. Setting up the relevant intervals, we find that

interval	$\left(0, \frac{1}{4a^2}\right)$	$\left(\frac{1}{4a^2}, \infty\right)$
sign of g'	positive	negative
property of g	increasing	decreasing

(The denominator of $g'(x)$ is always positive. When x is very close to 0, the quantity $1 - 2a\sqrt{x}$ is close to 1 and thus positive, and when x is large, the quantity $1 - 2a\sqrt{x}$ is negative.) Hence, the function g is increasing on the interval $\left[0, \frac{1}{4a^2}\right]$ and decreasing on the interval $\left[\frac{1}{4a^2}, \infty\right)$. Do not let the presence of the letter 'a' (often referred to as a parameter) rather than a number trouble you. The idea is that you are solving an entire collection of problems at the same time. ■

Extra Notes for Section 1.14

Example 1: Find all of the critical inputs and determine the nature of each one for the function f defined by $f(x) = (x^2 - 3x)^2(x-9)^3$.

Solution: As usual with such problems, we begin by finding and simplifying the derivative of the given function. In this particular case, patience and attention to detail are required. Using the product rule and the chain rule, we find that

$$\begin{aligned} f'(x) &= (x^2-3x)^2 \cdot 3(x-9)^2 + (x-9)^3 \cdot 2(x^2-3x)(2x-3) \\ &= (x^2-3x)(x-9)^2\big(3(x^2-3x) + 2(x-9)(2x-3)\big) \\ &= (x^2-3x)(x-9)^2\big(3x^2 - 9x + 4x^2 - 42x + 54\big) \\ &= (x^2-3x)(x-9)^2\big(7x^2 - 51x + 54\big) \\ &= (x^2-3x)(x-9)^2(x-6)(7x-9) \\ &= x(7x-9)(x-3)(x-6)(x-9)^2. \end{aligned}$$

The critical inputs occur at the five points 0, 9/7, 3, 6, and 9. We thus have six intervals to consider.

interval	$(-\infty, 0)$	$(0, 9/7)$	$(9/7, 3)$	$(3, 6)$	$(6, 9)$	$(9, \infty)$
sign of f'	$+$	$-$	$+$	$-$	$+$	$+$
prop of f	incr	decr	incr	decr	incr	incr

Applying the First Derivative Test to each of the critical inputs, we find that

1) there is a relative maximum value at $x = 0$;
2) there is a relative minimum value at $x = 9/7$;
3) there is a relative maximum value at $x = 3$;
4) there is a relative minimum value at $x = 6$;
5) there is no relative extreme value at $x = 9$.

It is certainly possible to use an electronic device to graph this function and look for turning points in the graph. However, be certain that you can do problems of this type without such devices. ■

Example 2: Let a be a positive constant and consider the function g defined by $g(x) = x + \dfrac{27a^4}{x^3}$. Find all of the critical inputs of g and determine the nature of each one.

Solution: Once again, we begin by finding and simplifying the derivative:

$$g'(x) = 1 - \frac{81a^4}{x^4} = \frac{x^4 - 81a^4}{x^4} = \frac{(x^2-9a^2)(x^2+9a^2)}{x^4} = \frac{(x-3a)(x+3a)(x^2+9a^2)}{x^4}.$$

The critical inputs occur at the points $\pm 3a$ (where $g'(x) = 0$) and also at 0 (where both $g(x)$ and $g'(x)$ are undefined). Checking the resulting four intervals, we find that

interval	$(-\infty,-3a)$	$(-3a,0)$	$(0,3a)$	$(3a,\infty)$
sign of g'	positive	negative	negative	positive
property of g	increasing	decreasing	decreasing	increasing

Hence, the function g is increasing on the intervals $(-\infty,-3a]$ and $[3a,\infty)$, and decreasing on the intervals $[-3a,0)$ and $(0,3a]$. We cannot conclude that g is decreasing on the interval $[-3a,3a]$ since g is not defined at 0. Even if g is defined at 0 by giving it some value, it is not true that g is decreasing on any interval containing 0. To see this, note that $g(x)$ is negative for $x<0$ and positive for $x>0$. Once again, it may be helpful to look at a graph of the function g for various values of the parameter a and verify the above conclusions. ■

Example 3: Find the maximum value for the product xy^2 given that x and y are positive numbers that satisfy the equation $x+4y=120$.

Solution: Solving the equation $x+4y=120$ for x gives us $x=120-4y$ so the quantity xy^2 becomes $(120-4y)y^2$. We thus want to find the maximum value of the function h defined by

$$h(y)=4(30y^2-y^3)$$

on the interval $(0,30)$. The interval is determined by the fact that both x and y are positive. In order for x to be positive, we need $y<30$. Taking the derivative yields

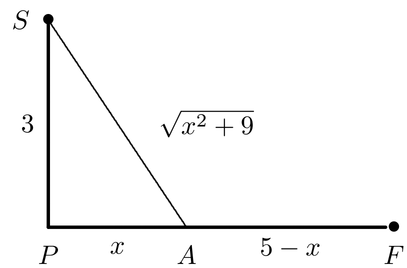
$$h'(y)=4(60y-3y^2)=12y(20-y)$$

and we see that the desired critical input is $y=20$. Noting that h' is positive on the interval $(0,20)$ and negative on the interval $(20,30)$, we confirm that h does have a maximum output at $y=20$. From the equation $x+4y=120$, when $y=20$, the value of x is 40. Hence, the maximum value for the product xy^2 is $40\cdot 20^2=16000$. ■

Extra Notes for Section 1.15

Example 1: A person in a boat three miles from shore wishes to reach a point five miles down the coast in the shortest time. Assuming that he can walk five miles per hour and row only four miles an hour, at what point must he land? (This problem appears in E. Nichols, *Differential and Integral Calculus with Applications*, D. C. Heath & Co., Boston, 1902.)

Solution: The problem is illustrated in the figure below. The point S is the starting position (in the water), the point F is the final destination (on the shoreline, which is implicitly assumed to be straight), and the point A is the place on the coast at which to direct the boat. Let P denote the point on the shore that is closest to S so that $\angle SPF$ is a right angle. Our goal is to find the distance x (that is, the length of PA) down the coast from P at which to aim the boat.



From the information in the problem, the length of SP is 3 miles and the length of AF is $5-x$ miles. It then follows that the person must row a distance of $\sqrt{x^2+9}$ miles and walk a distance of $5-x$ miles. Using the familiar fact that distance equals rate times time, the total time $T(x)$ for the trip is given by

$$T(x) = \frac{\sqrt{x^2+9}}{4} + \frac{5-x}{5}.$$

The relevant domain for this function is the interval $[0,5]$. We next perform the usual steps of taking the derivative and setting it equal to 0. Since (take the time to fill in any missing steps)

$$T'(x) = \frac{x}{4\sqrt{x^2+9}} - \frac{1}{5},$$

the equation $T'(x) = 0$ becomes

$$5x = 4\sqrt{x^2+9} \quad \Rightarrow \quad 25x^2 = 16(x^2+9) \quad \Rightarrow \quad 9x^2 = 16\cdot 9 \quad \Rightarrow \quad x = 4.$$

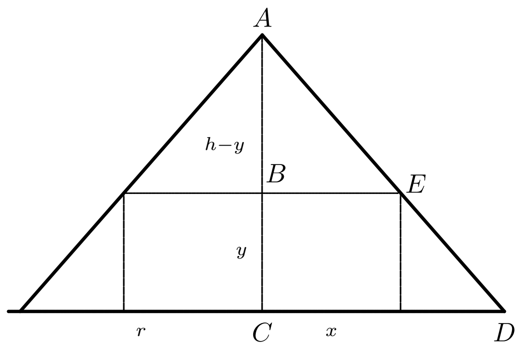
(Notice how the numbers are chosen so that the final answer comes out nice.) Since the endpoints in this case are reasonable options (row the minimum distance or row all of the way), we check the three outputs

$$T(0) = \frac{3}{4} + 1 = 1.75, \quad T(4) = \frac{5}{4} + \frac{1}{5} = 1.45, \quad T(5) = \frac{\sqrt{34}}{4} \approx 1.4577.$$

To minimize the total time, the person should direct the boat to a point four miles down the coast from the point P. However, note that rowing the entire way only takes about 30 seconds longer. It might be beneficial to ponder some of the assumptions that are needed to make this problem feasible. ■

Example 2: Determine the volume of the largest right circular cylinder that can be placed in a right circular cone of radius r and height h.

Solution: Rather than draw a three-dimensional picture of this problem, we show how the situation would appear if a slice through the centers of the cone and the cylinder were made. The black triangle represents the boundary of the cone and the green rectangle represents the boundary of the cylinder. We denote the radius and height of the cylinder by x and y, respectively.



We want to maximize the volume of the cylinder, that is, we want to make the quantity $\pi x^2 y$ as large as possible. To find a relationship between x and y, we use the fact that triangles $\triangle ABE$ and $\triangle ACD$ are similar to learn that

$$\frac{h-y}{h} = \frac{x}{r} \qquad \Leftrightarrow \qquad rh - ry = hx \qquad \Leftrightarrow \qquad y = \frac{h}{r}(r-x).$$

The volume V of the cylinder can now be represented as a function of x:

$$V(x) = \pi x^2 \cdot \frac{h}{r}(r-x) = \frac{\pi h}{r}(rx^2 - x^3),$$

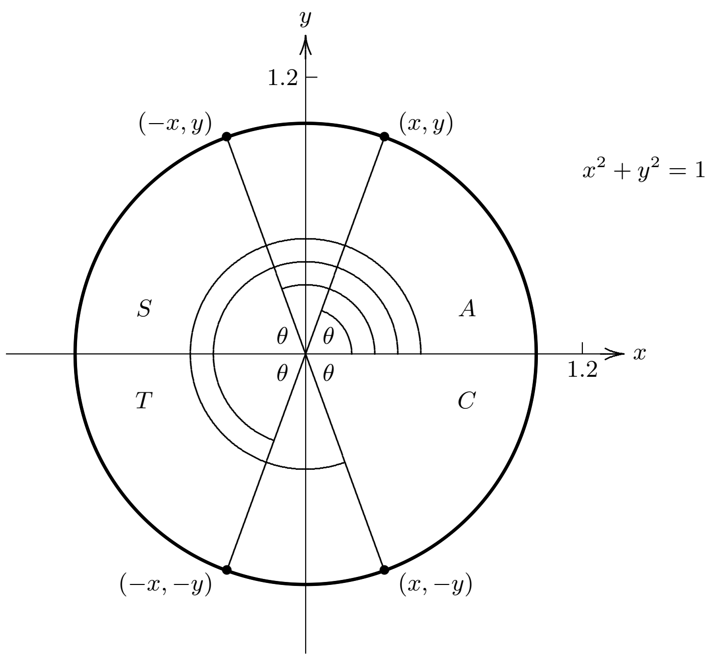
where $0 < x < r$. The equation $V'(x) = 0$ becomes $2rx = 3x^2$ so the desired value of x is $2r/3$. Rather than use the First Derivative Test in this case, we can appeal to the geometry of the situation to realize that this value must correspond to a maximum. The dimensions of the cylinder of maximum volume are thus

$$x = \frac{2r}{3} \quad \text{and} \quad y = \frac{h}{r}\Big(r - \frac{2r}{3}\Big) = \frac{h}{3} \quad \text{with a volume of} \quad V = \pi x^2 y = \pi\Big(\frac{2r}{3}\Big)^2 \cdot \frac{h}{3} = \frac{4}{9} \cdot \frac{1}{3}\,\pi r^2 h.$$

Since the volume of the cone is $\pi r^2 h/3$, we see that the cylinder of maximum volume occupies $4/9$ (which is less than half) of the entire cone. ■

Extra Notes for Section 1.16

We begin by making some comments concerning reference angles. Consider the graph of the unit circle given below, where the angle θ represents an acute angle based off the x-axis in each of the four quadrants.



In order of magnitude, the angles represented by the circular arcs are θ, $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$. The coordinates for the four highlighted bullet points are closely related due to the symmetry of the unit circle. As a consequence of the fact that the sine function corresponds to the y-coordinate and the cosine function corresponds to the x-coordinate, we find that

$$\begin{aligned} \sin(\pi - \theta) &= \sin\theta; & \cos(\pi - \theta) &= -\cos\theta; \\ \sin(\pi + \theta) &= -\sin\theta; & \cos(\pi + \theta) &= -\cos\theta; \\ \sin(2\pi - \theta) &= -\sin\theta; & \cos(2\pi - \theta) &= \cos\theta; \\ \sin(-\theta) &= -\sin\theta; & \cos(-\theta) &= \cos\theta. \end{aligned}$$

Hence, knowing the values of the trig functions in the first quadrant easily gives the values of the trig functions in the other three quadrants. The A, S, T, and C letters in the quadrants indicate that

A. all the trig functions are positive in Quadrant I;

S. the sine function is positive in Quadrant II;

T. the tangent function is positive in Quadrant III;

C. the cosine function is positive in Quadrant IV.

Some students remember this as All Students Take Calculus. The acute angle θ is often referred to as the reference angle.

Example 1: Find the exact values of all six trigonometric functions when the angle is $5\pi/3$.

Solution: The angle $5\pi/3$ lies in Quadrant IV and the reference angle is $\pi/3$. It follows that

$$\begin{aligned} \sin(5\pi/3) &= -\sin(\pi/3) = -\frac{\sqrt{3}}{2}; \\ \cos(5\pi/3) &= \cos(\pi/3) = \frac{1}{2}; \\ \tan(5\pi/3) &= -\tan(\pi/3) = -\sqrt{3}; \end{aligned} \qquad \text{and} \qquad \begin{aligned} \csc(5\pi/3) &= -\frac{2}{\sqrt{3}}; \\ \sec(5\pi/3) &= 2; \\ \cot(5\pi/3) &= -\frac{1}{\sqrt{3}}. \end{aligned}$$

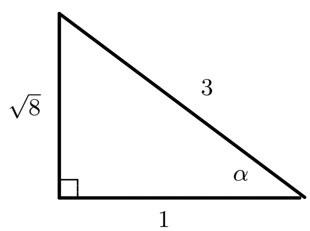
Example 2: Find the exact values of all six trigonometric functions when the angle is $5\pi/4$.

Solution: The angle $5\pi/4$ lies in Quadrant III and the reference angle is $\pi/4$. It follows that

$$\begin{aligned} \sin(5\pi/4) &= -\sin(\pi/4) = -\frac{\sqrt{2}}{2}; \\ \cos(5\pi/4) &= -\cos(\pi/4) = -\frac{\sqrt{2}}{2}; \\ \tan(5\pi/4) &= \tan(\pi/4) = 1; \end{aligned} \qquad \text{and} \qquad \begin{aligned} \csc(5\pi/4) &= -\sqrt{2}; \\ \sec(5\pi/4) &= -\sqrt{2}; \\ \cot(5\pi/4) &= 1. \end{aligned}$$

Example 3: Given that $\sec\theta = -3$ and $\pi < \theta < \frac{3}{2}\pi$, find the exact values of the other five trigonometric functions at θ.

Solution: We begin by considering an angle α for which $\sec\alpha = 3$, which is equivalent to $\cos\alpha = \frac{1}{3}$, and use a right triangle to express this fact.



Using the fact that the angle θ is in the third quadrant, it follows that

$$\begin{aligned} \sin\theta &= -\frac{\sqrt{8}}{3}; \\ \cos\theta &= -\frac{1}{3}; \\ \tan\theta &= \sqrt{8}; \end{aligned} \qquad \text{and} \qquad \begin{aligned} \csc\theta &= -\frac{3}{\sqrt{8}}; \\ \sec\theta &= -3; \\ \cot\theta &= \frac{1}{\sqrt{8}}. \end{aligned}$$

Example 4: Find all the values of x in the interval $[0, 2\pi]$ that satisfy $2\sin(2x) = 1$.

Solution: Letting $\theta = 2x$, this problem is equivalent to finding all of the values of θ in the interval $[0, 4\pi]$ that satisfy $\sin\theta = \frac{1}{2}$. In the interval $[0, 2\pi]$, the two values of θ are $\pi/6$ and $5\pi/6$. We can then add 2π to each of these values to get all of the solutions in the interval $[0, 4\pi]$, namely,

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}.$$

It follows that

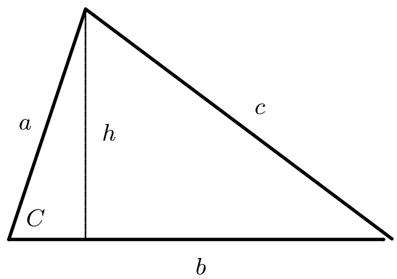
$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}.$$

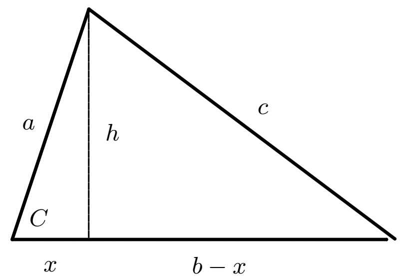
Example 5: Prove the law of cosines.

Solution: Given the terms listed in the left figure below, we want to prove that

$$c^2 = a^2 + b^2 - 2ab\cos C;$$

this is the law of cosines. (Note that this equation becomes the Pythagorean Theorem when $C = \pi/2$.)





Using the Pythagorean Theorem on the two right triangles in the figure on the right, we can represent h^2 in two different ways:

$$a^2 - x^2 = h^2 = c^2 - (b-x)^2.$$

It follows that

$$c^2 = a^2 - x^2 + (b-x)^2 = a^2 - x^2 + b^2 - 2bx + x^2 = a^2 + b^2 - 2ab\cos C,$$

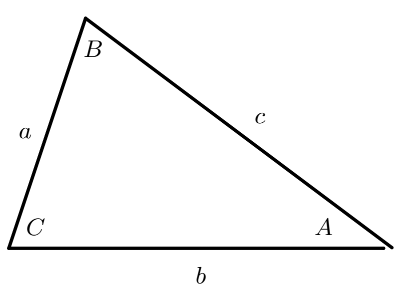
where we have used the fact $\cos C = x/a$.

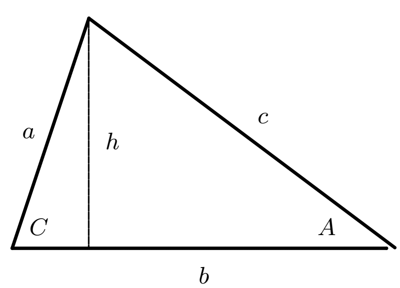
Example 6: Prove the law of sines.

Solution: Given the terms listed in the left figure below, we want to prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

this is the law of sines.





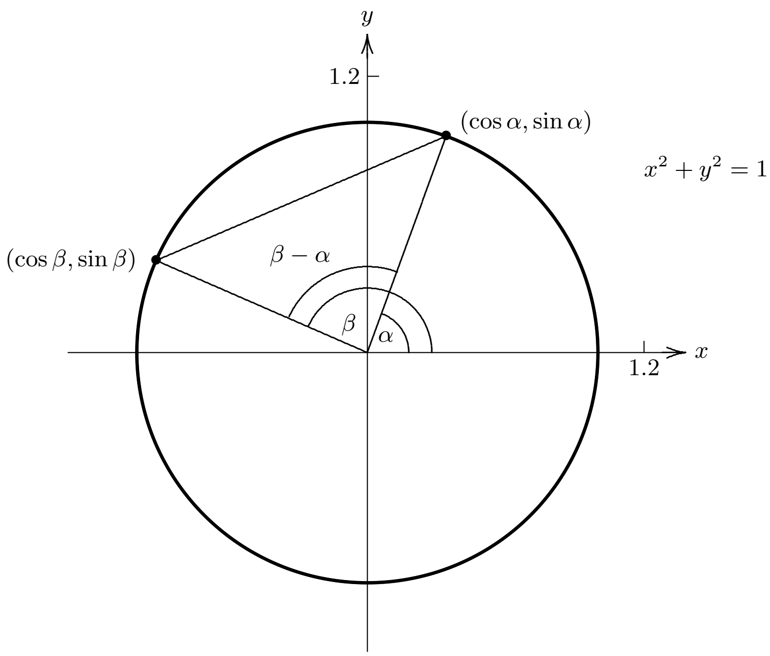
Referring to the figure on the right, note that $h = a\sin C$ and $h = c\sin A$. It follows that the area K of the triangle can be represented as

$$K = \frac{bh}{2} = \frac{ba\sin C}{2} \qquad \text{and} \qquad K = \frac{bh}{2} = \frac{bc\sin A}{2}.$$

Multiplying these two expressions by $2/b$ yields $a\sin C = c\sin A$ and thus $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$. The other equality involving b and $\sin B$ is proved in a similar way.

Example 7: Verify the trig identities for the sine and cosine of the sum and difference of two angles.

Solution: We consider the special case in which the angles α and β satisfy $0 < \alpha < \beta < \pi$; this generates a figure like the following.



Using the law of cosines on the red triangle (with the distance between two points formula) and noting the appearance of the most basic trig identity, we find that

$$\begin{aligned}
(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 &= 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos(\beta-\alpha);\\
\cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta - 2\sin\beta\sin\alpha + \sin^2\alpha &= 2 - 2\cos(\beta-\alpha);\\
2 - 2\cos\beta\cos\alpha - 2\sin\beta\sin\alpha &= 2 - 2\cos(\beta-\alpha);\\
\cos\beta\cos\alpha + \sin\beta\sin\alpha &= \cos(\beta-\alpha).
\end{aligned}$$

It then follows that

$$\cos(\beta+\alpha) = \cos(\beta-(-\alpha)) = \cos\beta\cos(-\alpha) + \sin\beta\sin(-\alpha) = \cos\beta\cos\alpha - \sin\beta\sin\alpha$$

and

$$\sin(\beta+\alpha) = \cos\Big(\frac{\pi}{2} - \beta - \alpha\Big) = \cos\Big(\frac{\pi}{2} - \beta\Big)\cos\alpha + \sin\Big(\frac{\pi}{2} - \beta\Big)\sin\alpha = \sin\beta\cos\alpha + \cos\beta\sin\alpha.$$

The last equation then yields

$$\sin(\beta-\alpha) = \sin\beta\cos(-\alpha) + \cos\beta\sin(-\alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha.$$

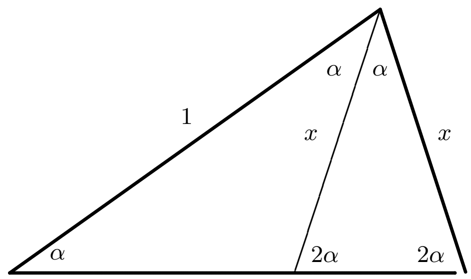
When $\alpha = \beta$, we find that

$$\cos(2\beta) = \cos^2\beta - \sin^2\beta \qquad \text{and} \qquad \sin(2\beta) = 2\sin\beta\cos\beta.$$

Many other trig identities then follow.

Example 8: In the list of simple values for sine and cosine, we have the angles π, $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$. It seems strange that the angle $\pi/5$ is omitted. Find the values of sine and cosine at $\pi/5$.

Solution: Let α represent the angle $\pi/5$, which is the same as $36°$. We want to find $\sin\alpha$ and $\cos\alpha$.



Referring to the figure above (in which three isosceles triangles appear), we notice that the whole triangle and the small triangle on the right are similar. It follows that

$$\frac{1-x}{x} = \frac{x}{1} \qquad \Leftrightarrow \qquad x^2 + x - 1 = 0 \qquad \Leftrightarrow \qquad x = \frac{-1 \pm \sqrt{5}}{2}.$$

Since x must be positive, we choose the root with the plus sign. Using the law of cosines on the whole triangle, we find that

$$x^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos\alpha \qquad \Leftrightarrow \qquad \cos\alpha = \frac{2 - x^2}{2} = \frac{2-(1-x)}{2} = \frac{1+x}{2} = \frac{1+\sqrt{5}}{4}.$$

(Make certain you follow each of these steps.) The number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \qquad \text{(which is the positive root of the equation } x^2 = x+1\text{)}$$

is an important constant in mathematics known as the golden mean or golden ratio. We have thus shown that

$$\cos(\pi/5) = \frac{\phi}{2} \qquad \text{and} \qquad \sin(\pi/5) = \sqrt{1 - \left(\frac{\phi}{2}\right)^2} = \sqrt{\frac{4-\phi^2}{4}} = \sqrt{\frac{3-\phi}{4}} = \frac{\sqrt{3-\phi}}{2}.$$

Using some simple trig identities, we then have

$$\cos^2(\pi/10) = \frac{1+\cos(\pi/5)}{2} = \frac{2+\phi}{4} \qquad \text{and} \qquad \sin^2(\pi/10) = \frac{1-\cos(\pi/5)}{2} = \frac{2-\phi}{4}.$$

We have thus shown that

$$\begin{aligned}
\sin(18°) &= \frac{\sqrt{2-\phi}}{2}; & \cos(18°) &= \frac{\sqrt{2+\phi}}{2}; \\
\sin(36°) &= \frac{\sqrt{3-\phi}}{2}; & \cos(36°) &= \frac{\phi}{2}; \\
\sin(54°) &= \frac{\phi}{2}; & \cos(54°) &= \frac{\sqrt{3-\phi}}{2}; \\
\sin(72°) &= \frac{\sqrt{2+\phi}}{2}; & \cos(72°) &= \frac{\sqrt{2-\phi}}{2}.
\end{aligned}$$

Hence, it is possible to find reasonably simple expressions for these values, but they are certainly more complicated than the values at the standard angles.