A bounded derivative that is not Riemann integrable

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3. There exist bounded derivatives that are not Riemann integrable.

A fangtooth fish: photo from National Geographic.

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If F is differentiable on $[a, b]$ and F' is bounded on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$. FALSE!

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We will present an example related to a collection of functions that Casper Goffman used in a March 1977 *Monthly* note.

Let $O = \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$ be an open dense set in the interval $(0, 1)$, where the intervals are disjoint and the inequality $\sum\limits_{n=1}^{\infty} (b_n - a_n) \leq \frac{1}{2}$ is satisfied. Without loss of generality, we may assume that $a_i \neq b_j$ for all positive integers i and j.

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There are various ways to construct such a set, each of which involve some prerequisite knowledge. The existence of such sets is far from obvious for students.

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For later reference, let $E = [0, 1] \setminus O$. Note that E is a perfect nowhere dense set of positive measure.

For each positive integer n, choose points c_n and d_n so that

$$a_n < c_n < d_n < b_n, \quad \frac{c_n + d_n}{2} = \frac{a_n + b_n}{2}, \quad \text{and} \quad d_n - c_n = (b_n - a_n)^2.$$

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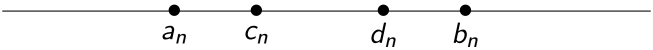

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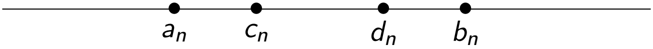

Note that

$$c_n - a_n = \frac{(b_n - a_n) - (b_n - a_n)^2}{2} = \frac{b_n - a_n}{2}\Big(1 - (b_n - a_n)\Big) > \frac{b_n - a_n}{4},$$

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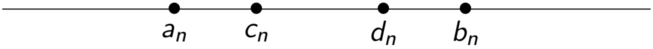

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and thus

$$(b_n - a_n)^2 < 16(c_n - a_n)^2.$$

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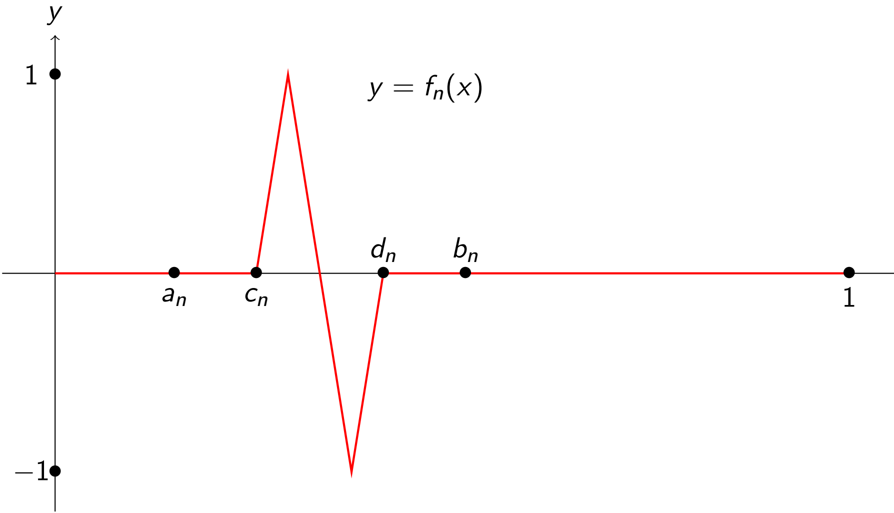
Define $f \colon [0, 1] \to [-1, 1]$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$.


y
1
−1
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a_n
c_n
d_n
b_n
1

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A bounded function f is Riemann integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exists a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that $\sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \epsilon$.

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If $i \in S$, then $(x_{i-1}, x_i) \subseteq O$.

If $i \in T$, then $(a_n, b_n) \subseteq (x_{i-1}, x_i)$ for some index n and thus $\omega(f, [x_{i-1}, x_i]) \ge 1$.

$$\sum_{i\in S}(x_i - x_{i-1}) \leq \sum_{n=1}^{\infty}(b_n - a_n) \leq \frac{1}{2} \quad \text{so} \quad \sum_{i\in T}(x_i - x_{i-1}) \geq \frac{1}{2}.$$

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$$\begin{aligned}\sum_{i=1}^{q}\omega(f,[x_{i-1},x_i])(x_i - x_{i-1}) &\ge \sum_{i\in T}\omega(f,[x_{i-1},x_i])(x_i - x_{i-1}) \\ &\ge \sum_{i\in T}(x_i - x_{i-1}) \ge \frac{1}{2}.\end{aligned}$$

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Hence, the function f is not Riemann integrable on $[0,1]$.

Now define a function F on $[0, 1]$ by

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We claim that $F'(x) = f(x)$ for all $x \in [0,1]$. It then follows that F' is a bounded derivative that is not Riemann integrable.

A fangtooth fish: photo from National Geographic.

First note that for all $x \in (a_n, b_n)$

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Hence $F'(x) = f_n(x) = f(x)$ for all $x \in (a_n, b_n)$ by the Fundamental Theorem of Calculus (the other version).

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photo of a yellowfish from the Christian Science Monitor

Now suppose that $x \in E$. We will prove that

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Otherwise fix a value of y so that $x < y < 1$. There are two cases to consider.

I. If $y \notin \bigcup_{k=1}^{\infty} (c_k, d_k)$, then $F(y) - F(x) = 0 - 0 = 0$.

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II. Suppose that $y \in (c_p, d_p)$. Using the inequality mentioned when the intervals $[c_n, d_n]$ were defined, we have

$$\begin{aligned} |F(y) - F(x)| = |F(y) - F(c_p)| &= \left| \int_{c_p}^{y} f_p \right| \\ &\leq \int_{c_p}^{y} |f_p| \leq d_p - c_p = (b_p - a_p)^2 \\ &\leq 16(c_p - a_p)^2 \\ &\leq 16(y - x)^2, \end{aligned}$$

where we have used the fact that $x < a_p < c_p < y$.

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We have thus shown that $F'(x) = 0$.

We conclude that $F'(x) = f(x)$ for all $x \in E$. It follows that $F' = f$ on $[0, 1]$.

Define sets V and W by

$$V = \{x \in [0,1] : f(x) \neq 0\} \subseteq \bigcup_{n=1}^{\infty} (c_n, d_n);$$
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The function f is approximately continuous at each point of E so f is approximately continuous on $[0,1]$.

Since the function F equals $\int_0^x f$, where the integral is a Lebesgue integral, we find that $F' = f$ on $[0,1]$ by a standard theorem in the theory of Lebesgue integration.

Thanks for listening.

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