Total: Name:

Math 244 **First Exam** **Fall 2011**

No electronic devices are allowed for this portion of the exam.

Write neat, concise, and accurate solutions to each of the following problems in the space provided. Include all of the relevant details and intermediate steps, with brief explanations as necessary, and conclude each problem with a complete sentence. Check your computations carefully. Each problem is worth 10 points.

1. The first problem concerns some of the theory of first order differential equations.

   a) Consider the initial value problem $(\cos t)y' + t^2 y = \ln t,\ y(1) = 8$. Find the largest open interval I on which the unique solution is guaranteed to exist.

-1 linear equation, put in standard form

$y' + \frac{t^2}{\cos t} y = \frac{\ln t}{\cos t}$ $\ln t$ cont on $(0, \infty)$

$\cos t = 0$ at odd multiples of $\pi/2$

$\frac{\ln t}{\cos t}$ cont on $(0, \frac{\pi}{2})$, $\frac{t^2}{\cos t}$ cont on $(-\frac{\pi}{2}, \frac{\pi}{2})$

since $t_0 = 1$, use $(0, \frac{\pi}{2})$

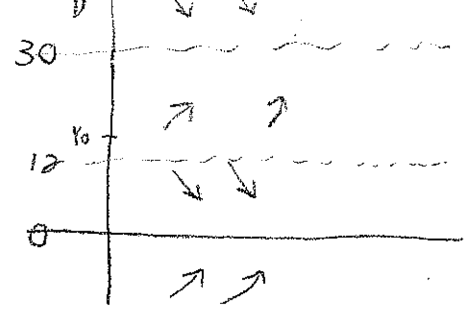
The solution to the IVP is guaranteed to exist on $(0, \frac{\pi}{2})$.

Let y be the solution to

   b) ~~Consider~~ the initial value problem $y' = 0.25\, y\,(12 - y)(y - 30),\ y(0) = 15$. Determine (with a brief explanation) $\lim_{t\to\infty} y(t)$.

use a direction field

equilibria are $y = 0$, $y = 12$, $y = 30$



From the direction field, it is clear that $\lim_{t \to \infty} y(t) = 30$.

2. Solve the initial value problem $y' = -\frac{1}{10}y^3$, $y(0) = \frac{1}{2}$ and find y explicitly. Then determine the value of y when $t = 25$.

separable equation, $\int \frac{dy}{y^3} = \int -\frac{1}{10}\,dt$

$-\frac{1}{2y^2} = -\frac{1}{10}t + c \Rightarrow \frac{1}{y^2} = \frac{1}{5}t + c_1$ $\qquad y(0) = \frac{1}{2} \Rightarrow 4 = c_1$

$\frac{1}{y^2} = \frac{1}{5}t + 4 = \frac{t+20}{5}$ or $y = \sqrt{\frac{5}{t+20}}$ $\qquad$ use $+\sqrt{\ }$ since $y(0) > 0$

The solution to the IVP is $y = \sqrt{\frac{5}{t+20}}$.

It follows that $y(25) = \frac{1}{3}$.

3. Find the general solution to the differential equation $y' = y + 2t - 7$.

linear equation

$y' - y = 2t - 7$ $\qquad I(t) = e^{-t}$ is the integrating factor

$(e^{-t}y)' = (2t-7)e^{-t}$ $\qquad$ use $\int t e^{kt}\,dt = \frac{1}{k^2}(kt-1)e^{kt}$

$e^{-t}y = \int (2te^{-t} - 7e^{-t})\,dt$

$= 2(-t-1)e^{-t} + 7e^{-t} + c$

$y = -2t - 2 + 7 + ce^{t}$

The general solution is $y = ce^{t} - 2t + 5$, where c is any constant.

$2t-7 \qquad e^{-t}$

$2 \qquad -e^{-t}$

$(-2t+7-2)e^{-t} + c$

4. A 500 gallon tank contains 400 gallons of fresh water. A brine containing one pound of salt per gallon of water runs into the tank at the rate of 5 gallons per minute, and the well-stirred mixture runs out of the tank at the same rate. Determine how long it takes for the amount of salt in the tank to reach one-half of its limiting value.

needed $A(t)$ lbs of salt at t min

$$A'(t) = 5\cdot 1 - 5\cdot\frac{A(t)}{400}, \quad A(0) = 0$$

using rate = rate in − rate out in lbs/min

$$A'(t) = 5 - \frac{1}{80}A(t)$$

$$\frac{d}{dt}(A(t) - 400) = -\frac{1}{80}(A(t) - 400)$$

$$A(t) - 400 = -400e^{-t/80}$$

using our basic result (linear, separable also)

$$A(t) = 400(1 - e^{-t/80})$$

The limiting value of salt is 400 lbs (using the direction field or common sense based on the inflow), so we want to solve $A(t) = 200$.

$$A(t) = 200 \Rightarrow 1 - e^{-t/80} = \frac{1}{2} \Rightarrow e^{t/80} = 2 \Rightarrow t = 80\ln 2$$

The amount of salt in the tank reaches half its

units limiting value after $80\ln 2$ minutes.

5. Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{3x^2 - 4y}{4x - 2y}$, $y(0) = -1$ explicitly for y.

rewrite the equation

$$(3x^2 - 4y)\,dx + (2y - 4x)\,dy = 0$$

-1

since $\frac{\partial}{\partial y}(3x^2 - 4y) = -4 = \frac{\partial}{\partial x}(2y - 4x)$, the equation is exact

we find that

+ 5 $\quad x^3 - 4xy + y^2 = C \qquad f_x(x,y) = 3x^2 - 4y$

has the desired properties $\qquad f_y(x,y) = 2y - 4x$

+ 1 $\quad y(0) = -1 \Rightarrow C = 1$

$$x^3 - 4xy + y^2 = 1$$

$y^2 - 4xy + 4x^2 = 1 - x^3 + 4x^2$ complete the square

+ 2 $\quad (y - 2x)^2 = -x^3 + 4x^2 + 1$

$+2$

$y = 2x - \sqrt{-x^3 + 4x^2 + 1}$ use $-$ sign since $y(0) = -1$

The solution to the IVP is $y = 2x - \sqrt{-x^3 + 4x^2 + 1}$.

I certify that the solutions written below are my own work. ______________

Name: *Signature*

Math 244 **First Exam (Continued)** **Fall 2011**

Write a neat, concise, and accurate solution for the problem on this page; partial credit will be kept to a minimum. Include all relevant details, make certain all of the steps are clear, use correct notation, and use complete sentences when appropriate. I strongly recommend that you work the problem first on a separate page then copy a more polished solution onto this page (use the space wisely). You are expected to work independently on this test (remember that blue plagiarism form you signed as well as the statement above) but you may use the textbook and your notes. This problem is due by 5 pm on Thursday (9/22); either slide it under my office door or put it in my Olin Hall mailbox.

6. A college graduate has a debt of \$30,000. If the annual interest rate is 5% and she makes payments at a rate of $4000 - 2400e^{-t/10}$ dollars per year after t years, determine the number of years it will take her to pay off the loan and how much interest she pays on the loan. (Assume, as we have been doing, that the payments are made continuously and that the interest accrues continuously.) You will need the assistance of some electronic device to determine the numerical values requested in this problem; please identify the type of device you used.

✓

-1 $A(t)$ dollars owed after t years

from the given data, we have

$$A'(t) = \frac{1}{20}A(t) - \left(4000 - 2400e^{-t/10}\right), \quad A(0) = 30000$$

-1 the diff eq is linear — integrating factor

$$A'(t) - \frac{1}{20}A(t) = 2400e^{-t/10} - 4000 \qquad I(t) = e^{-t/20}$$

$$\frac{d}{dt}\left(e^{-t/20}A(t)\right) = 2400e^{-3t/20} - 4000e^{-t/20}$$

$$e^{-t/20}A(t) = -16000e^{-3t/20} + 80000e^{-t/20} - 34000$$

where we have used the initial condition to find the constant of integration

$$A(t) = 80000 - 34000e^{t/20} - 16000e^{-t/10}$$

solve $A(t) = 0$ to determine the length of the loan

using Maple, we find that $t = 16.315(06292)$ call this value x

The loan is paid off after about 16.3 years.

-1 The total payments are $\int_0^x (4000 - 2400e^{-t/10})\,dt$ and Maple

gives a value of 45955.48 (full x) / 45955.26 (3 decimal x)

$4000x + 24000(e^{-x/10} - 1)$

$4000(x - 6 + 6e^{-x/10})$

-1 The amount of interest paid on the loan is about \$15955.

A(t) 5

t 2

int 3

Exam

59, 59

6
55, 55, 55
54, 54

8

5, 45, 45
44
43
42
41
40
39
38
37, 37, 37
36, 36

34

29

27

24

15

52 - 60	8
44 - 51	5
36 - 43	11
28 - 35	2
0 - 27	7

HW

36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36
35, 35, 35, 35
34, 34
33, 33
32
31

29

28

26

19

16

12

32 - 36	27
28 - 31	3
24 - 27	1
20 - 23	0
0 - 19	3